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INSTRUMENTAL ANALYSIS*

BY VANNEVAR BUSH

The use of instruments of computation and analysis is as old as mathematics itself. Counting by the aid of piles of pebbles gave origin to the word calculus. There have always been such aids to reasoning, and the development of instrumental methods has throughout proceeded in parallel with formal methods.

Under instrumental analysis is to be grouped all analysis proceeding by the use of devices for supplementing pure reasoning, whether these devices involve mechanics, optics, heat, electricity or other natural phenomena. The device aids the mind because it approximately obeys some simple law, and may be made to indicate the consequences of combinations of such relationships. An example is the ordinary compass which will draw an approximate circle. One may reason about the properties of families of circles and never draw a diagram; but such reasoning is more surely guided by a few pictures. The instrument, whatever it may be, has two functions: first, when approximate results are sufficient, to yield these directly; second, as a suggestive auxiliary to precise reasoning.

The first piece of mathematical apparatus used was the human hand. Nature, for some reason that is not entirely clear, supplied us with ten fingers, and in the groping development that characterized the beginnings of mathematics, this settled the decimal system upon us. Our forebears apparently did their reckoning standing up, or we might, as in fact did more than one tribe, have used toes as well, and thus landed on the vicenary system. Had this occurred it would have been possible to register all of New York's automobiles with five figures; and Congress would probably have had to be satisfied with merely eight figure appropriations. It is certainly fortunate that at least both hands were used, or the newspapers would now be running out of zeros.

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How Archimedes managed to approximate π as well as he did by an involved geometrical computation in Greek numerals has not been adequately explained. The world suffered long for lack of positional numeration, and for a symbol for zero. Neither of these were invented by formalists. Both were the product of instrumental analysis. They came as the direct and inevitable result of the use of the abacus.

To the occidental it is sometimes necessary to explain that the abacus is the most widely used mechanical computing machine, even today. An expert in its use can nearly keep up with the operator of a simple keyboard-adding-machine, and he can multiply as well. It is a very old device, and no one knows its inventor. Originally, probably, a group of piles of pebbles, it developed into a form in which counters are strung on parallel rods or wires. The mechanical fact that it is convenient to mount rods or wires parallel to one another in a frame produced the idea of positional numeration, and the necessity for noting down complete absence of counters under such circumstances gave us the zero. That the world had to wait so long for these important ideas occurred because the formalists of the time declined to use the plebeian abacus in connection with their profound meditations. So they stuck to their cumbersome notation, while men of trade, with a mechanical aid, produced the most far-reaching of mathematical inventions. The formalists insisted that ruler and compass were the only tools worthy of the gentleman scholar, and by this insistence directed the attention of the learned for centuries to three impossible problems, made so by the artificiality of the limitations. Even today the race of angle trisectors has not died out, although the problem was resolved over a century ago.

Thus, from the earliest times, and in the most profound ways, the use of instruments has influenced the course of formal mathematics. The fixing of the decimal system by the possession of ten fingers, the origination of positional numeration and the zero by the users of the abacus, the cramping of the mathematical style of the Greeks by the ruler and compasses, are incidents in the process. The instrument has been much more than the useful aid; it has often exerted a determining control over the course of events, helpfully or otherwise. This situation is not likely to terminate, and in fact there is every reason to expect

that the interrelation will be more complex, if not more intimate, now that mechanical devices may employ a host of new elements and processes. Yet there is a considerable gulf between formal mathematicians and those who make and use mathematical instruments. It is the purpose of this paper to suggest that formal mathematics may again be influenced by instrumental development, and that there would be mutual benefit if the gulf between them were less wide.

When Babbage struggled to put into effect his glorious ideas of mechanical analysis, he was stopped by the expense and delay inherent in his time in the construction of a really complex device. Today much more complex affairs are built at reasonable cost, by reason of mass production of duplicate parts, modern gauging and materials, and modern processes of fabrication. Moreover, and very important, these complex devices are *reliable*; witness the automatic telephone switchboard or the typesetting machine. This development has had a large effect in the field of arithmetical computation. It will have a comparable effect, ultimately, in the fields of other mathematical instruments. There is, nevertheless, a serious barrier to be overcome before this occurs. Reliability comes, in a complicated machine, only when a great deal of study and experiment is devoted to the design of individual parts, which are then fabricated by methods that produce large numbers of precise replicas at low unit cost. The spread between development and production cost may be enormous: thousands of dollars may be spent in perfecting a simple relay or lever which may later be produced for a few cents each, provided hundreds of thousands are made. This barrier militates heavily against the research tool, which is potentially useful in only a few laboratories. Otherwise there would be available a much greater variety of mathematical instruments than at present, and their performance would be more satisfactory. Yet it appears, in spite of this inherent limitation, that we are at the beginning of an important period of development of machines for higher analysis.

The numerical machine is old; Pascal made an excellent adding machine of which we fortunately have records [1],* and

* Numbers in square brackets refer to the bibliography at the end of this paper.

Leibnitz one capable of performing the four operations of arithmetic [2]. These early devices involved all of the really basic inventions necessary for the successful construction of machines of this class. Leibnitz performed multiplication by repeated addition, shifting columns as the several digits were treated, and this procedure persists in most modern machines. It remained for Bollé to introduce the multiplication table in the form of a three-dimensional cam, and multiplication is thus performed by fewer operations, as many as there are digits in the multiplier [3]. This of course gave his machine greater speed, and facilitated division as well. It is comparatively recently that such machines can be built at moderate cost, and sufficiently reliable and rugged to withstand continuous hard usage. The combination of such machines with punched cards [4], [5] has made arithmetic into an entirely new affair, and it has enabled large banks, insurance companies, and businesses generally to operate in ways that would be fantastic by longhand methods. There is a great deal more arithmetic and better arithmetic in the world than there used to be. This is indicated by the fact that 10,000 tons of cards are used per year, a total of four billion cards or nearly one card per dollar of a famous recent appropriation. The end of the development is not in sight, and much remains to be accomplished. Part of this is due to the fact that the development of such machines is highly expensive, so that they have been produced to meet the needs of large markets, namely, those of the usual business. The usual business doesn't go far beyond rather simple mathematical operations. Yet, now that there is momentum, the large possibilities of the process will undoubtedly be opened up.

The future development of arithmetical devices would appear to lie along the lines of their employment for more complex calculations than at present. One matter that needs attention is an increase in the amount of information that can be placed on a card. The obvious way, that of decreasing the size of holes, has its limitations, although if photography and photo-cell controls were used a great deal could be thus accomplished. A factor of 1000 in the amount of information which may be stored in a given volume appears thus readily attainable. We now have two main types of punched-card apparatus, that operated by making electrical contacts through the holes, and that operated

by mechanical rods which pass through the holes to actual controls. These may later be joined by a type in which a light beam passes through and operates a photo-cell. This opens up large vistas, for the light may be pulsating and made to produce a control depending upon its frequency or duration.

There is another way of increasing the information on a card, that of using combinations. This is just beginning to be used. The usual card is divided into columns, commonly 40 or 80 in number. There are 10 positions in a column. One hole only is punched in a column, and by its vertical position indicates a number. Thus only 80 numbers may be carried by a card. If these numbers are typed by typewriter, the space they occupy is less than 1/10 of the area of the card necessary to carry them by the punched scheme. This is not good space economy. To improve it, combinations are used. Two holes, producing a joint effect, may together designate a number and shorten the column. This is readily extended to letters, and thus 80 letters may be placed on a standard card. However this principle of combinations might be carried much farther. Using combinations of two, five positions are sufficient for 10 numbers, thus doubling the card capacity. Carrying the idea to extremes, although with evident difficulty of practical utilization, much more can be attained. There are 800 positions on a card. Combinations of these by two's allow over 300,000 bits of information to be carried. This would allow roughly 10,000 letters of the alphabet, and their sequence, to be carried on a single card, enough to record the information on three typed sheets of usual size. The use of combinations of holes may evidently be much extended.

Great ingenuity has been shown in obtaining desired sequences of operations; but there is still a great deal of carrying cards from one machine to another, and each problem is unique. A group of columns on a card assigned to a particular number, or set of symbols, is called a field. A stack of cards constitutes a two-dimensional array of fields, and each field may contain a number, to a precision determined by the size of field chosen. Apparatus is available for automatically combining any two such numbers, thus recorded on a given card or adjacent cards, in accordance with any of the four numerical operations, and recording the result in a similar manner. The process of bring-

ing the correct numbers into position for the performance of operations, especially for a long sequence of operations, is now the laborious procedure. It involves carrying the cards from one to another of many machines, and waiting each time for the specific operation to be completed. A master control is here conceivable. This process should also be automatic and performed entirely by machine. It should be sufficient, having punched a stack of cards, then to punch a master card, dictating with complete flexibility the operations to be performed, their sequence, the pairs of numbers in the array upon which the operations are to be impressed, and the position in the array where the results are to be joined to the array. A numerical process, however complex, would then be reduced to the recording of the raw data, and the exact specification by similar record of the desired numerical process, all else being relegated, as it should be, to the machine. Such an arrangement would no doubt be soon worked out if there were sufficient commercial demand. This would be a close approach to Babbage's large conception [4] as far as arithmetical processes are concerned. It would complete, for arithmetic, the consummation which Leibnitz visualized for all mathematics.

Quite a lot can be accomplished by arithmetic alone, if its operations can be performed rapidly enough, and combined with suitable facility. Witness the difference engine, projected by Babbage, and suffering inevitably from the high cost of precision construction of the time. Feed one of these machines the skeleton of a table, and it will fill in and print the entire table. It has been used for this purpose to good effect. It consists essentially of adding machines coupled together so that a whole sequence of operations is automatically performed one after another, corresponding in this case to the steps involved in higher order interpolation. Complete machines of this sort have been built by Scheutz and others [6]. It is time that a numerical machine were built for which the sequence of operations might be varied at will to cover a large field of utility, but just as fully automatic once the sequence is assigned.

There is not the slightest doubt that numerical machines are destined to be further developed and exceedingly useful. Their influence on the users of arithmetic has been powerful. Business

does not seem to realize as yet that it needs anything more than arithmetic in its ordinary affairs, and so other devices dealing with branches of mathematics beyond arithmetic have not had the benefit of aggressive commercial development. They are none the less interesting and important, and ultimately their influence may be as great or greater. They will serve, however, the scientist and research man rather than the man of business, at least at first.

No applied mathematics except arithmetic is ever exact, and even arithmetic often merely has that appearance. So when we turn from arithmetic we find devices that are expected to yield approximations to mathematical processes, sufficiently precise for use in applications. There are two main classes, the operators and the equation solvers. No short summary can be at all complete in regard to either, for the literature of the subject is enormous, and each device requires much exposition if it is to be all understood. This is especially true since there is often such a wide gap between the conception of such an intrument and its actual construction and use, so that details are often important. Hence only a few will be selected for illustration in what follows, with full regret that every interesting development cannot be given its due recognition.

Before a mathematical operation can be performed mechanically by an instrument of the first class, the data must appear in the form of a mechanical record. For this purpose two sorts of records are necessary, those representing numbers, and those representing functions. On the form of record depends the means that may be utilized in performing the operation.

Numbers may be recorded in many ways. The position of holes in a card has proved a very fruitful way. The most obvious is by means of a distance between index lines. This leads directly to the combinations of such distances by geometric means, including link motions. The slide rule of Oughtred [7], the alignment charts or nomograms of d'Ocagne, follow almost inevitably. Some of the modern counterparts of these are highly complex and are likely to be genuinely useful only when the occasion for use is sufficiently extensive to submerge the not inconsiderable time necessary for their mastery. The systematization of nomograms, by identification with determinants [7] has

been especially helpful in this regard, for it replaces chance invention and sometimes bizarre manipulation to a standardized affair.

Usually the geometrical combination of distances representing numbers is limited to cases where moderate precision is ample. The much coiled slide rules become rather cumbersome. That this is not an inherent limitation is shown however, by Harrison's machine for evaluating differences [8]. This machine was developed for rapidly and precisely recording the enormous array of differences between wave numbers involved in the analysis of spectra. The numbers are represented by distances to holes punched in a long tape which is folded back on itself. A light above makes records through the holes on a sheet of sensitized paper moving slowly below in a direction perpendicular to that of the tape. The position of the dots then gives the desired differences directly. A long loop insures high precision.

Anything measurable may of course be used to represent a number. Thus the deflection of a spring is especially useful when many products are to be obtained. This was employed, for example, in the Michelson-Stratton harmonic analyser [9], which operated with a finite number of ordinates or components, so that it was a device for combining sets of numbers in predetermined fashion.

The representation of a function is most readily made by means of a simple graph. Once represented, the operations of differentiation and integration are evidently fundamental. There have been many ways of performing the latter, but very few of the former, and this is due to the fact that precision in taking the derivative of a function approximately represented by a curve immediately involves all of the difficulty that would be mathematically expected. Moreover many devices projected for this purpose have attempted the impossible feat of finding a derivative by examining one point. Some, such as the optical devices, have depended upon the ability of the eye to detect readily a discontinuity in the derivative of a curve [10]. It would be interesting to examine the extent to which the eye is able to note discontinuities, in fact there seems to be some connection between this matter and the artist's idea

of a pleasing curve, but neither the psychologists nor the artists seem to have investigated the subject systematically.

The most reasonable way to obtain a derivative mechanically appears to be to measure the average slope over an adjustable known interval, and this Sears does very neatly [11]. The curve is made into an optical mask, which may be of either the variable area or the variable density type, and this is oscillated in the direction of abscissas in front of a slit. Light passing the slit is summed by a photo-electric cell. The fundamental component of the alternating photocell current is then a measure of the derivative. Both the width of slit and amplitude of oscillation are adjustable, to adapt to varying degrees of precision in the curve.

One of the easiest ways to take a derivative is electrically. If the form of a varying current through a pure inductance represents a function, the voltage across the inductance represents the derivative. Unfortunately it is not easy either to cause a current to vary precisely in a prescribed manner, nor to measure precisely a varying voltage. The combination of mask, optical system, and photo-cell will accomplish the current control. This has appeared, combining sinusoids, in acoustical devices for electrical organs, and of course in the ubiquitous talkies. None of these applications require the precision desirable in mathematical instruments. A varying voltage may be very conveniently measured by modern cathode-ray oscillographs, but again not with the precision desirable for purposes of analysis.

The integral of a function represented by a varying current may be obtained by passing the current through a condenser, and measuring the resulting voltage as before. We shall return to this matter of electrical-circuit methods later.

When a function is represented by a graph, the most obvious way to obtain an integral is to cut the curve out and weigh it. This yields a definite integral, and not a running integral, as does in fact the simplest planimeter. Planimeters are noteworthy as instruments which are mechanically simple and mathematically complex. The extreme is the old knife-edge or hatchet planimeter of Prytz [12], which works surprisingly well, and causes so much involvement when one is called upon to explain why.

Cutting out a mask to represent a function, uniformly illuminating it, and collecting the light that goes through on a photo-cell, gives an integral conveniently, but not highly precisely on account of the limitations of photo-cells and optical systems. It does, however, give a possibility of interesting combinations, which will be mentioned later.

Any adjustable-speed drive may be made into an integrator. If the setting of the adjustment of the ratio of drive is made to vary in accordance with a given function of a variable, while one shaft turns in accordance with the variable itself, the rotation of the output shaft gives the integral. If the shafts are interchanged the output then gives the integral of the reciprocal of the function. James Thomson's disc-ball-and-cylinder integrator was of this type [13]. The ball contacts the disc at a point on a diameter, and the cylinder at a point on an element, and transmits motion between them. The disc is turned in accordance with the independent variable, the displacement of the ball from the center is regulated in accordance with the function to be integrated, and the rotation of the cylinder yields the integral. Its modern form, in the hands of Hannibal Ford, has become a rugged and reliable instrument [14]. Ford's form has two balls, with direct pressure, and the presence of two balls allows displacement under pure rolling. Maxwell gave a discussion of this type of integrator [13], but unfortunately not a complete analysis. If one assumes no backlash, no plastic deformation, and no slippage, the performance is exact; but in a physical instrument the areas of contact are finite. One point only of the area is a point of zero relative motion, and this point shifts in the area when there is any torque, due either to friction or inertia.

With high pressure on the balls considerable load may however be carried and with remarkable precision, although care must be taken in regard to mechanical hysteresis.

For highly precise work the disc and roller type [15] appears preferable, for here the area of contact is greatly reduced, and indeterminateness of the position of the zero point is rendered less important. The edge of the roller is hard and fairly sharp, and its shaft is actually in jewel bearings. Such a unit is capable of excellent performance, but it can carry practically no load. Hence, to be useful, it can merely control rotation, the work

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1935—H. S. Vandiver.
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1923—M. I. Pupin.
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1927—E. W. Brown.
1928—G. H. Hardy.
1929—Irving Fisher.
1930—E. B. Wilson.
1931—P. W. Bridgman.
1932—R. C. Tolman.
1934—Albert Einstein.
1935—Vannevar Bush.

Visiting Lecturers:

1927-28—Constantin Carathéodory, University of Munich.
1928-29—Hermann Weyl, Zurich Technical School.
1929-30—Enrico Bompiani, University of Rome.
1930-31—Wilhelm Blaschke, University of Hamburg.
1931-32—R. L. Moore, University of Texas.
1936-37—Thirukkannapuram Vijayaraghavan, University of Dacca.

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E. R. Hedrick, 1929-1930,
L. P. Eisenhart, 1931-1932,
A. B. Coble, 1933-1934.

ENDOWMENT FUND

In 1923 an Endowment Fund was collected for the support of the ever increasing number of important mathematical memoirs. Of this fund, which now amounts to some $77,000, a considerable proportion was contributed by members of the Society.

PRIZE FUNDS

The Bôcher Memorial Prize.

This prize was founded in memory of Professor Maxime Bôcher. It is awarded every five years for a notable research memoir in analysis which has appeared during the preceding five years in a recognized journal published in the United States or Canada; the recipient must be a member of the Society, and not more than fifty years old at the time of publication of his memoir.

First (Preliminary) Award, 1923: To G. D. Birkhoff, for his memoir *Dynamical systems with two degrees of freedom.*

Second Award, 1924: To E. T. Bell, for his memoir *Arithmetical paraphrases,* and to Solomon Lefschetz, for his memoir *On certain numerical invariants with applications to abelian varieties.*

Third Award, 1928: To J. W. Alexander, for his memoir *Combinatorial analysis situs.*

Fourth Award, 1932: To Marston Morse, for his memoir *The foundations of a theory of the calculus of variations in the large in m-space,* and to Norbert Wiener, for his memoir *Tauberian theorems.*

The Frank Nelson Cole Prize in Algebra.
The Frank Nelson Cole Prize in the Theory of Numbers.

These prizes were founded in honor of Professor Frank Nelson Cole, on the occasion of his retirement as Secretary of the American Mathematical Society and editor of the *Bulletin,* after twenty-five years of service; the fund was later doubled by his son, Charles A. Cole. They are to be awarded at five-year intervals for contributions to algebra and the theory of numbers, respectively, under restrictions similar to those for the Bôcher prize.

First Award, 1928: To L. E. Dickson, for his book *Algebren und ihre Zahlentheorie,* Zurich, 1927.

Second Award, 1931: To H. S. Vandiver, for his several papers on Fermat's last theorem published in the *Transactions of the American Mathematical Society* and the *Annals of Mathematics* during the last five years, with special reference to a paper entitled *On Fermat's last theorem,* which appeared in volume 31 of the *Transactions.*

The Eliakim Hastings Moore Fund.

This fund was founded in 1922 in honor of Professor Eliakim Hastings Moore, on the occasion of the twenty-fifth anniversary meeting of the Chicago Section of the American Mathematical Society. The interest on the fund is to be used at the discretion of the Council of the Society, for the publication of important mathematical books or memoirs or the award of prizes.

HISTORICAL NOTE

On November 24, 1888, six persons connected with Columbia University, four with the Department of Mathematics and two with the Department of Astronomy, formed a society to meet monthly for the discussion of mathematical topics. This society was the idea of Doctor T. S. Fiske, an instructor in mathematics, who had recently returned from Cambridge, England, where he had studied for a time with Cayley, Glaisher, and Forsyth. In Glaisher's company he had attended several meetings of the London Mathematical Society; from this fact and from a desire for mathematical companionship, stimulated by his experiences in Cambridge and especially by Glaisher's many friendly talks about mathematics and mathematicians, arose his eagerness to bring about the establishment of a society in New York.

A month later at the second meeting of the society, it was resolved to adopt the name "New York Mathematical Society" and to invite to membership everyone living in or near New York who might be interested. The young society prospered, its meetings interested those who attended them, and its membership gradually increased. In 1889 its activities were greatly quickened through the accession of Doctor Emory McClintock, who was called to New York as Actuary of the Mutual Life Insurance Company. He strengthened the Society not only by his interest in scientific research but also by his administrative wisdom. At the annual meeting in December, 1890, he was elected president in succession to Professor Van Amringe of Columbia University.

In December, 1890, it was proposed that the Society publish a journal which should contain reports of its meetings, historical and critical articles, general mathematical news, and short original papers. Doctor McClintock pointed out that in order to put this proposal into effect it would be necessary greatly to enlarge the membership of the Society; and at once the Secretary undertook to send to mathematicians all over the country a prospectus of the proposed journal and an invitation to join the Society. In June, 1891, the membership of the Society had increased to 174, and in October of the same year the first number of the *Bulletin of the New York Mathematical Society* made its appearance.

By the beginning of 1894 it was generally recognized that the Society had assumed a national character; and in response to a demand that came from many quarters its name was changed to the American Mathematical Society. A few months later the Society voted to provide funds for the publication of the papers read at the International Mathematical Congress held in Chicago in connection with the World's Fair in 1893. This was the first book published by the Society. The same year the Society held its first summer meeting in connection with the Brooklyn meeting of the American Association for the Advancement of Science.

In 1895, Professor Cole was called from the University of Michigan to Columbia University and became Secretary of the Society, in which capacity he continued to serve the Society for twenty-five years, being succeeded by the present Secretary in 1920.

In 1896, at the suggestion of Professor H. S. White, the Society arranged for a colloquium to be held in connection with the summer meeting at Buffalo; since then, at intervals of one, two, or three years, nineteen colloquia have been held. Beginning with that held in Boston in 1903 the lectures have been published in book form.

To provide for regular meetings at strategic points not on the Atlantic seaboard various sections were established from time to time. In 1897 there was organized the Chicago Section, in 1902 the San Francisco Section, and in 1907 the Southwestern Section. These sections have now all been merged into the parent body. The total number of regular meetings held is now 334, besides sectional meetings. From the outset, the field of the Society has included Canada; important meetings have been held in Toronto in the summer of 1897 in connection with the British Association for the Advancement of Science and in the winter of 1921 with the American Association for the Advancement of Science.

In April, 1899, the Society felt that the time had come to inaugurate a journal which should contain the more important original papers presented at its meetings. This resulted in the publication of the *Transactions of the American Mathematical Society* beginning in 1900 with Professors E. H. Moore, E. W. Brown, and T. S. Fiske as editors. The initial financial difficulties were overcome through the generosity of a number of universities which granted subventions annually during a period of several years.

In 1899, Professor Fiske, who had served for nine years as chief editor of the *Bulletin*, retired in favor of Professor Cole, who after twenty-one years of service was succeeded in 1920 by Professor Hedrick.

Beginning in 1901 the Society entered into an agreement with Columbia University whereby that institution catalogues its library and makes the arrangements necessary for the loan of its books; in return the University makes use of the Society's collection as a reference library. The most recent catalogue of the library issued by the Society is under date of 1932. From the very beginning, Columbia University has been liberal in its hospitality. Not only have half of the meetings of the Society been held in its buildings, but it has provided more than its share of officers and has furnished space to house its activities. The present quarters are dignified and commodious.

In order to take over problems concerned primarily with teaching and organization of college mathematics and with relations to secondary school instruction, a new organization, the Mathematical Association of America, was founded in 1915. Nearly all the members of the Society are members of this new body and the organizations work together cordially, generally meeting in conjunction.

At the outbreak of the war in 1914, the Society had reached a membership of about 700 and was recognized as one of the great mathematical societies of the world. During that conflict it was so fortunate as to be able to hold its ground but could hope for no growth. At the end of this difficult period, Professor Cole retired in 1920, after distinguished service rendered for twenty-five years as secretary and for twenty-one years as chief editor of the *Bulletin*.

It became plain that the war had left many serious problems and that there was a crisis in the financial affairs of the Society. The great increase in the cost of mathematical printing that occurred during and after the war made it necessary for the Society to enlarge its resources if its publications were to be continued. Accordingly steps were taken to enlarge the membership, with the result that the latter has grown from 750 in 1920 to approximately 1850 in 1936. Through the devoted services of a small group in the Society an Endowment Fund* was collected. At one time more than thirty Sustaining Members, comprising some of the great engineering firms and insurance companies of the country, as well as several universities, contributed annually to the support of its activities. The General Education Board and the Rockefeller Foundation have for the period 1925-36 granted generous and substantial subventions to aid in printing the journals. During several years special contributions were solicited from members.

In the past fifteen years, the activities of the Society have been considerably extended. The Josiah Willard Gibbs Lectureship, established in 1923, furnishes to the general scientific public (by means of an annual lecture) an opportunity for appreciation of mathematics and its applications. The Society Visiting Lectureship, the incumbent of which is a distinguished foreign or American mathematician, opens a channel for many universities to hear each year a lecture or series of lectures on some mathematical topic of current interest. Colloquia are now held yearly instead of triennially, with a corresponding increase in the number of volumes printed; prizes** have been established; reciprocity agreements have been arranged with the London Mathematical Society, the Deutsche Mathematiker-Vereinigung, and the Unione Matematica Italiana by each of which the members of one organization enjoy special privileges in the other. And the library has been strengthened.

* See page 6.
** See page 6.

Beginning with 1927 the Society entered into a contract with the Johns Hopkins University by which it shares in the responsibility for the *American Journal of Mathematics;* this agreement, which increased the printing space available by four or five hundred pages, solved for the moment the problem of publication facilities confronting American mathematics. A further most welcome contribution in this direction was the founding in 1935 of the Duke Mathematical Journal by Duke University.

The number of papers read before the various meetings has grown to aggregate approximately four hundred annually. In spite of rising standards of acceptance, the demand for space in both the *Bulletin* and the *Transactions* has increased so that each prints approximately one thousand pages, the establishment by the Rockefeller Foundation of National Research Fellowships in mathematics being in part responsible for this increase.

The financial depression raised problems for the Society comparable in their seriousness with those of a dozen years earlier. The number of Sustaining Members fell to less than one-half and, beginning with July, 1936, the subvention from the Rockefeller groups was discontinued. Active steps were taken in this crisis to increase the revenue in other directions lest there be a disastrous curtailment of the activities of the Society. New types of membership were established in which the minimum payments are considerably in excess of the regular dues. Professor M. H. Ingraham, with the support of the Rockefeller Foundation, spent the year 1934-35 in interesting the universities and colleges of the country in the support of mathematical publications. This campaign, which was markedly successful, resulted in the enlisting of several score additional institutional members. There are now more than 80 institutions, with total annual dues of approximately $7,000.

There is still a demand for increasing the channels for publication of research, and the establishment of a journal of applied mathematics has been suggested.

In 1923 the organization was incorporated under the code of the District of Columbia. It is proposed that the Society celebrate in 1938 the fiftieth anniversary of its founding, and plans for a worthy program are well under way. It is hoped that at that time a history of the activities of the Society can be published.

STATISTICS

Year	*Members*	*Papers Presented*	*Cost of Publications*	*Total Expenditure*
1895	260	24	$ 556	$ 862
1900	347	112	2,900	3,346
1905	488	147	4,100	5,200
1910	630	145	3,300	5,000
1915	721	197	3,000	4,900
1917	732	167	4,300	6,000
1919	724	187	3,500	5,200
1921	770	175	5,200	8,300
1923	1,250	266	11,600	15,600
1925	1,542	315	11,000	15,000
1927	1,758	404	24,000	28,000
1928	1,796	439	22,000	27,000
1929	1,926	395	18,000	25,000
1930	1,926	386	25,000	31,000
1931	1,883	374	25,500	32,000
1932	1,882	321	29,000	36,000
1933	1,783	415	23,000	30,000
1934	1,782	343	26,500	34,000
1935	1,863	436	23,000	35,000

INSTITUTIONAL MEMBERS

SEPTEMBER, 1936

COLLEGES, UNIVERSITIES, UNIVERSITY GROUPS, AND LEARNED SOCIETIES

Amherst College, Amherst, Mass.
Beloit College, Beloit, Wis.
Bowdoin College, Brunswick, Me.
Brigham Young University, Provo, Utah.
Department of Mathematics, Brooklyn College, Brooklyn, N.Y.
Brown University, Providence, R.I.
Bryn Mawr College, Bryn Mawr, Pa.
California Institute of Technology, Pasadena, Calif.
Carnegie Institute of Technology, Pittsburgh, Pa.
Case School of Applied Science, Cleveland, Ohio.
Catholic University of America, Washington, D.C.
Members of the Department of Mathematics, College of the City of New York, New York, N.Y.
The College of St. Thomas, St. Paul, Minn.
Colorado College, Colorado Springs, Colo.
Columbia University, New York, N.Y.
Connecticut College, New London, Conn.
Cornell University, Ithaca, N.Y.
Dartmouth College, Hanover, N.H.
Duke University, Durham, N.C.
Georgetown University, Washington, D.C.
Gettysburg College, Gettysburg, Pa.
Division of Mathematics, Harvard University, Cambridge, Mass.
Haverford College, Haverford, Pa.
Indiana University, Bloomington, Ind.
Institute for Advanced Study, Princeton, N.J.
Iowa State College, Ames, Iowa.
The Johns Hopkins University, Baltimore, Md.
Louisiana State University, Baton Rouge, La.
McGill University, Montreal, Quebec, Canada.
Massachusetts Institute of Technology, Cambridge, Mass.
Mathematical Association of America, Oberlin, Ohio.
Michigan State College of Agriculture and Applied Science, East Lansing, Mich.
New York University, New York, N.Y.
Northwestern University, Evanston, Ill.
Ohio State University, Columbus, Ohio.
General Research Council of the State of Oregon, Corvallis and Eugene, Ore.
Pennsylvania State College, State College, Pa.
Library of Purdue University, Lafayette, Ind.
Rutgers University, New Brunswick, N.J.
Smith College, Northampton, Mass.
Stanford University, Stanford University, Calif.
The State University of Iowa, Iowa City, Iowa.
Stevens Institute of Technology, Hoboken, N.J.
Department of Mathematics, Swarthmore College, Swarthmore, Pa.
Syracuse University, Syracuse, N.Y.
Tulane University, New Orleans, La.
University of Buffalo, Buffalo, N.Y.
University of California, Berkeley, Calif.
University of California at Los Angeles, Los Angeles, Calif.
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University of Cincinnati, Cincinnati, Ohio.
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University of Florida, Gainesville, Fla.
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University of Kansas, Lawrence, Kan.
Department of Mathematics, University of Kentucky, Lexington, Ky.

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University of Michigan, Ann Arbor, Mich.
Department of Mathematics, University of Nebraska, Lincoln, Neb.
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University of North Carolina, Chapel Hill, N.C.
Department of Mathematics, University of Oklahoma, Norman, Okla.
University of Pennsylvania, Philadelphia, Pa.
University of Pittsburgh, Pittsburgh, Pa.
University of Rochester, Rochester, N.Y.
University of Southern California, Los Angeles, Calif.
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University of Virginia, Charlottesville, Va.
University of Washington, Seattle, Wash.
University of Wisconsin, Madison, Wis.
Virginia Polytechnic Institute, Blacksburg, Va.
Department of Mathematics, Washington University, St. Louis, Mo.
Department of Mathematics, Wayne University, Detroit, Mich.
Department of Mathematics, Wellesley College, Wellesley, Mass.
Wells College, Aurora, N.Y.
Wesleyan University, Middletown, Conn.
Western Reserve University, Cleveland, Ohio.
Department of Mathematics, Williams College, Williamstown, Mass.
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Equitable Life Insurance Company of Iowa, Des Moines, Iowa.
John Hancock Mutual Life Insurance Company, Boston, Mass.
Metropolitan Life Insurance Company, New York, N.Y.
Prudential Insurance Company, Newark, N.J.
Western and Southern Life Insurance Company, Cincinnati, Ohio.

INSTITUTIONS FINANCIALLY COOPERATING WITHOUT TECHNICALLY BECOMING MEMBERS

Lehigh University, Bethlehem, Pa.
National Research Council, Washington, D.C.
Princeton University, Princeton, N.J.
Rockefeller Foundation, New York, N.Y.

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O. J. Peterson
Frankfort
W. W. Jones
Lexington
Allison
P. P. Boyd
L. W. Cohen
H. H. Downing
Jenkins
F. John
Latimer
LeStourgeon
Pence
Louisville
Bullitt
Stevenson
Murray
Carman
Nazareth
Mary Charlotte (Fowler)
Richmond
R. S. Park

LOUISIANA

Baton Rouge
Parker
Rutt
Sanders
H. L. Smith
New Orleans
Newcomb College
N. L. Anderson
M. G. Humphreys
Many
M. C. Spencer
Tulane University
H. E. Buchanan
Comfort
Duren
C. Hopkins
J. F. Thomson

MAINE

Brunswick
Hammond
Holmes
Korgen
W. A. Moody
Orono
Bryan
S. H. Kimball
Waterville
Schoenberg

MARYLAND

Aberdeen
Dederick
Annapolis
Ball
Bramble
Capron
R. Church
Clements
A. E. Currier
Kells
Littauer
T. W. Moore
Rawlins
Root
Scarborough
L. T. Wilson
Baltimore
Dawkins
Dorroh
L. B. Robinson
Sutton
Goucher College
C. L. Bacon
F. P. Lewis
Torrey
Johns Hopkins University
A. Cohen
J. H. Curtiss
R. H. Fox
Harry
Haviland
van Kampen
Kershner
F. Morley
Morrill
Murnaghan
L. J. Reed
Trott
Turpin
J. Williamson
Wintner
Zariski
Cheltenham
Hartwell
Chestertown
W. J. Robinson
College Park
Dantzig
Frederick
R. G. Mason
Westminster
Spicer

MASSACHUSETTS

Amherst
B. L. Brown
C. W. Cobb
Esty
Sprague
Belmont
C. W. Franklin
R. R. Struik
Boston
Cleary
Dysart
A. B. Gould
L. A. McCoy
Pousland
Rines
Boston University
Bruce
Mode
Graduate School of Business Administration, Harvard University
T. H. Brown
Leavens
Harvard School of Public Health
E. B. Wilson
John Hancock Mutual Life Insurance Company
Abbott
H. A. Garabedian
Grout
Howe
Vatter
Northeastern University
E. E. Haskins
Spear
Simmons College
Hemenway
Brockton
N. Kaplan
Brookline
Groat
A. L. Miller
Cambridge
Galbraith
Korzybski
E. W. Wilson
Harvard University
Ahlfors
Andrews
Beatley
G. Birkhoff
G. D. Birkhoff
Boas
Chafee
H. E. Clifford
Coolidge
Crum
W. C. Graustein
Huntington
R. F. Jackson
Kennelly
W. T. Lewis
Lyman
MacNeille
Mosesson
Oxtoby
Pitcher
Shapley
H. M. Sheffer
M. H. Stone
Street
J. H. Van Vleck
Walsh
H. Whitney
Widder
Massachusetts Institute of Technology
V. Bush
Cameron
Crout
Douglas
Douglass
P. Franklin
C. E. Fuller
Goodwin
Harvey
Hopf
Keyes
W. T. Martin
Phillips
Rutledge
D. J. Struik
W. C. Taylor
Vallarta
Wadsworth
Wiener
F. S. Woods
Zeldin
Radcliffe College
M. C. Graustein
Chestnut Hill
McCormick
Marcou
Dorchester
Kales
M. M. Sullivan
Ipswich
Kinsman
Lynn
Bauer
Medford
M. S. Barnes
Tufts College
J. L. Barnes
Ransom
F. G. Wren
Natick
R. Willis
Newton Highlands
Holley
Northampton
S. R. Benedict
N. H. McCoy
D. Montgomery
Rambo
R. G. Wood
North Cambridge
Osgood
Norton
C. A. Garabedian
Petersham
Moriarty
Southbridge
Boeder
South Hadley
F. E. Baker
Doak
Litzinger
Springfield
A. V. Newton
W. B. Robinson
Swampscott
G. W. Evans
E. Thomson
Wellesley
Clara E. Smith
Wellesley College
L. P. Copeland
H. A. Merrill
Russell
Stark
M. M. Young
Weston
Leonarda (Burke)
West Somerville
Schaeffer
Williamstown
Agard
J. G. Hardy
D. E. Richmond
Shepard
V. H. Wells
Worcester
Rabinow
A. H. Wheeler
Worcester Polytechnic Institute
Gay
R. K. Morley
H. Rice

MICHIGAN

Albion
Sleight
Ann Arbor
Grennan

University of Michigan
W. L. Ayres
Baten
Bradshaw
Churchill
A. H. Copeland
C. C. Craig
Dushnik
Dwyer
Field
W. B. Ford
Glover
T. H. Hildebrandt
L. A. Hopkins
Kaltenborn
Karpinski
Langford
Laporte
C. E. Love
MacKay
Menge
E. W. Miller
Nyswander
V. C. Poor
Rainich
Running
Timoshenko
R. W. Wagner
Wehausen
Welmers
R. L. Wilder

Detroit
Denton
Fisk

Detroit Institute of Technology
Goldman

University of Detroit
McCarthy
Muehlmann

Wayne University
Borgman
Coral
Fischer
Folley
Morrow
A. L. Nelson
H. H. Pixley

East Lansing
P. L. Dressel
V. G. Grove
J. D. Hill
W. S. Kimball
H. L. Olson
Plant
Powell
Speeker

Ferndale
Abernethy

Grand Rapids
A. Wilson

Hillsdale
Beeler

Kalamazoo
Mary Christopher
Walton

Marquette
Spooner

Nazareth
Fites

Royal Oak
Ladue

MINNESOTA

Coleraine
Fattu

Minneapolis
Jensen
Varino

University of Minnesota
Brink
Brooke
Bussey
E. Carlson
Dalaker
Gibbens
Gunstad
W. L. Hart
D. Jackson
Ness
Sagen
Scammon
Scherberg
Shumway
Underhill
Wegner
Wilcox

Moorhead
M. R. Anderson
E. J. Olson

Northfield
M. B. White

Oak Terrace
M. A. Wilder

Rochester
Oberg

St. Paul
Palmer

College of St. Thomas
L. E. Bush
Nosker
Thielman

Winona
Leopold

MISSISSIPPI

Agricultural College
C. D. Smith

Alcorn
W. B. Nelson

Cleveland
Hickey

Ellisville
McFarland

Gulfport
O'Ferrall

Jackson
D. McCoy
Sampson

State College
Ollivier

University
Hume

MISSOURI

Clayton
Rosskopf
Rule

Columbia

Christian College
Spalding

University of Missouri
H. Betz
Blumenthal
H. C. Carter
Ewing
Wahlin
Westfall

Jefferson City
Talbot

Kansas City
Basye
Melcher

University of Kansas City
Luby
Sigley

Lexington
Munro

Parkville
Crull

St. Louis
E. L. Harris
Herzog
E. W. King
Rafferty
R. L. Short

Harris Teachers College
Barr
Osborn

Maryville College
Marie (Kernaghan)

St. Louis College of Pharmacy
DeFoe

St. Louis University
Case
Larguier
Regan
A. E. Ross
Vezeau

Washington University
Buell
Butter
Dunkel
Langsdorf
Rider
Roever
J. Y. Stephens
Szegö

Springfield
Finkel
H'Doubler

Warrensburg
Scarborough

MONTANA

Bozeman
Hurst

Havre
W. I. Foster

Helena
Canning
Topel

Missoula
E. F. A. Carey
Lennes
A. S. Merrill

NEBRASKA

Hastings
Westhafer

Lincoln

Union College
Specht

University of Nebraska
Basoco
Brenke
C. C. Camp
Candy
Doole
Gaba
Pierce
Runge
Stafford
Webster

Omaha
Earl

Wayne
J. W. Boyce
Hove

NEVADA

Reno
H. C. Ayres
F. Wood

NEW HAMPSHIRE

Concord
G. M. Conwell

Durham
Slobin
Solt
Exeter
Pennell
Hanover
E. P. Brown
Morgan
Dartmouth College
Beetle
B. H. Brown
Forsyth
C. N. Haskins
Mathewson
F. W. Perkins
R. Robinson
Silverman
C. E. Wilder
Stratham
Wiggin

NEW JERSEY

Belleplain
Durell
Elizabeth
Eason
Hackettstown
LaMotte
Hoboken
Appuhn
L. E. Armstrong
H. N. Davis
Deimel
Hazeltine
Linden
D. L. Fuller
Mountain Lake
Shewhart
Newark
Conkling
L. L. Jackson
E. E. Rhodes
J. S. Thompson
Widmark
Essex County Junior College
Bull
Newark College of Engineering
Fithian
Prudential Insurance Company
Bartleson
Brush
Fitzgerald
V. Howell
Kineki
Lundgren
Sosnow
New Brunswick
New Jersey College for Women
Meder
C. A. Nelson
Walter
Rutgers University
Brasefield
Bunyan
H. S. Grant
R. Morris
Starke
Orange
Strock
Perth Amboy
Fullerton
Princeton
Burton
Flood
Parke
R. C. Stauffer
Wallman
Institute for Advanced Study
Alexander
Baer
A. H. Clifford
Comenetz
Fialkow
M. Hall
M. Kline
R. S. Martin
M. Morse
Myers
von Neumann
G. B. Price
M. Richardson
Tompkins
Veblen
Weyl
Zippin
Princeton University
E. P. Adams
Allendoerfer
Bochner
Bohnenblust
A. Church
Dribin
Eisenhart
Gillespie
Knebelman
Lefschetz
H. P. Robertson
Steenrod
T. Y. Thomas
Titt
A. W. Tucker
Wedderburn
Wigner
Wilks
Willson
Roselle
Huntley
Lepeshkin
South Orange
Rauch
Stanwick
Trenton
Shuster
Upper Montclair
G. A. Campbell
New Jersey State Teachers College
D. R. Davis
Fehr
E. H. C. Hildebrandt
Mallory
West Orange
Edison

NEW MEXICO

Albuquerque
Barnhart
Newsom
Las Vegas
Rodgers
Roswell
Alden
Santa Fe
Mickelson
Silver City
P. M. Singer
State College
P. K. Rees

NEW YORK

Albany
Glasgow
B. F. Kimball
W. R. Thompson
New York State College for Teachers
Beaver
Birchenough
DoBell
Lester
Shover
E. C. Stokes
Aurora
Holcroft
E. C. Rusk
Brook Haven
Hawkins
Brooklyn
E. M. Blake
Deutsch
DeVita
Haight
M. Halperin
Hamermesh
Langman
Lassar
L. L. Locke
Maddaus
Schuyler
Brooklyn College
Borofsky
Boyer
Dillon
Feld
Fleisher
H. M. Griffin
R. A. Johnson
Kennison
J. P. Kormes
A. W. Landers
MacNeish
L. T. Moore
Nicholson
Prenowitz
J. Singer
J. Wolfe
Woodbridge
Long Island University
Knobelauch
H. G. Lieber
L. R. Lieber
Polytechnic Institute of Brooklyn
William Johnston Berry
Fenn
Greeley
Whitford
Pratt Institute
Cowles
E. D. Smith
J. E. Thompson
St. Joseph's College for Women
Francis Xavier
Buffalo
Podmele
Sheridan
D'Youville College
Mary Imelda
University of Buffalo
Gehman
Montague
E. R. Ott
Pound
Clinton
H. S. Brown
Carruth
Ferry
Cohocton
Mehlenbacher
Crown Point
R. Henderson
Elmira
Suffa
F. M. Wright
Garden City
J. Bowden
Geneva
Durfee
Hubbs
Groton
C. E. Rhodes

Hamilton
Aude
A. W. Smith
Wardwell
Highland Mills
H. G. Kaplan
Ithaca
E. A. Miller
Cornell University
Agnew
W. B. Campbell
Carver
Clark
Flexner
Hurwitz
B. W. Jones
Karapetoff
Lawrence
D. C. Lewis
Lowenstein
MacLane
Randolph
J. B. Rosser
Sharpe
V. Snyder
Tanner
Trevor
R. J. Walker
Warschawski
Jamaica
Shiffman
Long Island City
Morrison
Mt. St. Vincent-on-Hudson
Catharine Marie
New York
J. Alsberg
J. W. Arnold
Berkeley
Berman
M. A. Bowden
R. W. Burgess
Clingan
W. B. Coleman
Cooper
Ellis
Frankel
Gallatin
Gibbs
W. J. Graham
Greitzer
S. E. Hotelling
Jablonower
Jacobus
Joffe
Kagno
S. Y. Keyser
F. G. King
Kircher
M. Kormes
Laderman
Lazarus
Mirick
Moir
Murphy
Neuman
Otten
Parsons
E. A. Porter
Preston
C. A. Richmond
Robbins
Roosevelt
L. G. Simon
D. E. Smith
Tanzola
Warner
W. Weaver
A. W. Whitney
E. C. Williams
W. J. Willis
American Mathematical Society
E. M. Hull
Odoms
American Telephone and Telegraph Company
Bagnall
Belcher
J. R. Carson
R. M. Foster
Halbert
Hoyt
Mead
Bell Telephone Laboratories
Bode
Dietzold
Fry
Gray
P. C. Jones
Kammerer
MacColl
Molina
Olmstead
Riordan
Schelkunoff
Sunde
Vaage
Wise
Zobel
College of the City of New York
Boeker
Carrie
Echols
B. P. Gill
C. C. Grove
Hubert
Ingram
Linehan
Nathan
Philip
Post
F. G. Reynolds
S. L. Robinson
R. C. Shook
R. F. Smith
Wetzel
Wirth
H. N. Wright
College of the Sacred Heart
Mira
Columbia University
R. G. Archibald
A. C. Berry
A. B. Brown
M. A. Durand
A. A. Evans
Fiske
Fite
Gourin
Hawkes
Hofmann
H. Hotelling
Kasner
C. J. Keyser
Koopman
Lorch
Mullins
F. J. Murray
Northcott
Pfeiffer
C. L. Poor
Reeve
Ritt
Ruger
Serghiesco
Siceloff
P. A. Smith
Thorndike
Upton
H. B. Williams
Wills
Cooper Union
F. H. Miller
Reddick
Hunter College
R. L. Anderson
Becker
Blanch
Bradley
Jewell H. Bushey
J. H. Bushey
Eisele
Guggenbühl
L. S. Hill
M. K. Landers
Levin
McLaughlin
M. S. Rees
Simons
Weisner
Metropolitan Life Insurance Company
Bassford
William James Berry
R. V. Carpenter
Christman
J. D. Craig
Hohaus
Lotka
Penney
Mutual Life Insurance Company
Monsky
Strong
New York University
Adler
Baron
F. Bernstein
Cooley
Courant
Farnum
Flanders
P. H. Graham
F. C. Hall
Helliwell
F. W. John
Kittell
McKinsey
Payne
A. S. Peters
R. G. Putnam
S. G. Roth
Schlauch
Tilley
Wahlert
Yanosik
Yeshiva College
Ginsburg
Lowan
Niagara University
Banks
Ogdensburg
Vaughan
Oneonta
Harrison
Schoonmaker
Perry
G. W. Walker
Poughkeepsie
Cowley
L. D. Cummings
Hopper
G. Smith
Weiss
M. E. Wells
H. S. White
Rochester
W. Betz
Herzberger
University of Rochester
Gale
Leighton

Long
Seidel
Watkeys
Rockville Center
Van Dyck
St. Bonaventure
Colbert
O'Brien
Saratoga Springs
E. N. Akeley
Schenectady
General Electric Company
Alger
Poritsky
Ruiz
Union College
Burkett
Farrell
A. H. Fox
Garis
D. S. Morse
A. D. Snyder
Ulrich
Vedder
Scotia
Leibing
Syracuse
Metzler
Roe
Syracuse University
A. D. Campbell
I. S. Carroll
Decker
Harwood
Troy
E. B. Allen
D. B. Ames
Cosby
McGiffert
West Point
H. Jones
Willseyville
Seely

NORTH CAROLINA

Chapel Hill
Browne
Garner
A. Henderson
Hobbs
Kattsoff
Lasley
Linker
Mackie
Winsor
Clinton
Culbreth
Davidson
W. W. Wood
Denver
Nixon
Durham
Carlitz
F. G. Dressel
W. W. Elliott
Gergen
Greenwood
Hickson
E. R. C. Miles
Patterson
Roberts
J. M. Thomas
Greensboro
Daniels
Pegram
Greenville
ReBarker
Maxton
Key
Raleigh
Bullock
Cell
J. M. Clarkson
Levine
H. V. Park
Salisbury
Dearborn
Statesville
R. W. Stokes

NORTH DAKOTA

Fargo
I. W. Smith

OHIO

Alliance
Hildner
Ashland
Rinehart
Athens
Marquis
F. W. Reed
Starcher
Bluffton
Hirschler
Bowling Green
Ogg
Overman
Cincinnati
University of Cincinnati
Barnett
Brand
Farnau
Hancock
Kersten
Lubin
Mendel
Merriman
H. L. Miller
C. N. Moore
P. M. Pepper
Szász
Yowell
Western and Southern Life Insurance Company
Biscay
Chenoweth
Learson
Ryan
Stilwell
Xavier University
J. F. Butler
Cleveland
Case School of Applied Science
O. E. Brown
Burington
Focke
D. C. Miller
M. Morris
Nassau
C. F. Thomas
Torrance
Fenn College
Tolar
Ursuline College
Mary Angela (Margraf)
Western Reserve University
M. G. Boyce
Musselman
Sauté
W. G. Simon
Columbus
J. C. Rietz
Ohio State University
Bamforth
Bareis
Blumberg
Hartung
Kuhn
LaPaz
Manson
Radó
Rasor
Toops
J. H. Weaver
Wylie
Dayton
Midgley
Delaware
Rowland
Gambier
R. B. Allen
Bumer
Granville
Kato
Ladner
Wiley
Hiram
Clarke
Kent
Brooks
F. Harshbarger
Manchester
Stelson
Loudonville
Feinler
Marietta
A. H. Black
Cope
Mt. St. Joseph
Maria (Corona)
North Canton
Schug
Oberlin
W. D. Cairns
Carr
M. M. Johnson
Yeaton
Oxford
Miami University
W. E. Anderson
Pollard
Western College for Women
Tappan
South Euclid
Mary Cleophas (Garvin)
Toledo
Brandeberry
Dancer
Wilberforce
Davidson
Wilmington
Spinks
Wooster
C. O. Williamson
Yanney
Yellow Springs
Astrachan

OKLAHOMA

Bartlesville
Hagler
Edmond
L. W. Johnson
Holdenville
Wedel
Norman
La Fon
University of Oklahoma
Brixey
Court
Hassler
Meacham
Reaves
Stillwater
Gundersen

OREGON

Corvallis
McAlister
W. E. Milne
Van Groos

Eugene
DeCou
Ghent
Moursund
Portland
Reed College
F. L. Griffin
J. M. Short
University of Portland
Godfrey (Vassallo)

PENNSYLVANIA

Allentown
Cedar Crest College for Women
Hallett
Muhlenberg College
Deck
Koehler
Annville
P. S. Wagner
Bethlehem
Beale
S. S. Cairns
R. N. Cobb
Cutler
Fort
Lamson
Latshaw
D. H. Lehmer
Raynor
C. A. Shook
Smail
Vanderslice
Bryn Mawr
Comegys
T. S. Peterson
Steinmetz
Bryn Mawr College
Hedlund
Lehr
Tuller
A. P. Wheeler
California
Bernstorf
Carlisle
F. Ayres
Landis
Chambersburg
Davison
R. F. Johnson
Weeks
Chester
F. G. Williams
Clarion
C. E. Carey
Collegeville
Clawson
Easton
R. P. Bailey
Beverley
W. M. Smith
East Pittsburgh
Slepian
Erie
R. E. Smith
George School
Pieters
Gettysburg
Arms
Greensburg
Marie Gertrude (McNeil)
Harrisburg
Whited
Haverford
Gummere
Oakley
A. H. Wilson
Immaculata
Mary Columba
Jenkintown
J. C. Durand
Kutztown
Knedler
Latrobe
Yahner
Lewisburg
Drum
New Wilmington
H. L. Black
Philadelphia
Lawton
Linton
McDonough
Routon
Sherman
V. E. Spencer
Winston
Drexel Institute
Tartler
Temple University
Gleason
University of Pennsylvania
Babb
Beal
Caris
J. A. Clarkson
H. B. Evans
W. L. Graves
J. R. Kline
Lufkin
Lyle
Milgram
H. H. Mitchell
Rademacher
Safford
Shohat
Shugert
A. J. Smith
Witmer
Pittsburgh
Holly
Allegheny Observatory
Burns
Carnegie Institute of Technology
Dines
Hicks
R. P. Johnson
Morehead
Moskovitz
Neelley
Olds
Riggs
Rosenbach
Saibel
Serbin
T. L. Smith
Stoker
Whitman
Pennsylvania College for Women
Calkins
University of Pittsburgh
Culver
Foraker
Knipp
Speer
J. S. Taylor
Scranton
Mary Bertrand (Walton)
State College
A. H. Frink
Jeffries
H. B. Owens
Pennsylvania State College
T. Cohen
Curry
O. Frink
Gordon
Gravatt
C. H. Graves
Hagen
Krall
E. I. Moody
F. W. Owens
Rupp
I. M. Sheffer
Stecker
C. C. Wagner
West
Swarthmore
Swann
Swarthmore College
Brinkmann
Dresden
Marriott
Uniontown
E. Johnson
Wallingford
J. A. Miller
Washington
Atchison
Dorwart
Shaub
Wayne
Makarov
Wilkes-Barre
W. I. Miller
York
Schweigert

PHILIPPINE ISLANDS

Manila
Artiaga
Gokhale

PORTO RICO

Mayaguez
Morales
Rio Piedras
Horne
Kuschke

RHODE ISLAND

Providence
R. B. Adams
Brown University
C. R. Adams
R. C. Archibald
A. A. Bennett
D. L. Bernstein
Carlen
Coon
C. H. Currier
Eberhart
Frame
Gilman
Lindsay
H. P. Manning
Middleswart
Rawhouser
R. G. D. Richardson
R. M. Robinson
Royall
C. H. Smiley
Tamarkin
Tukey
West Warwick
Brant

SOUTH CAROLINA

Charleston
J. Anderson
Dye
Clinton
H. E. Spencer
Columbia
J. B. Coleman
J. B. Jackson

Orangeburg
Jewell
Rock Hill
A. A. Grant

SOUTH DAKOTA

Brookings
Walder
Wyckoff
Vermillion
H. I. Lane
Yankton
E. J. Camp

TENNESSEE

Brownsville
Maclin
Jackson
Farnsworth
Michel
Knoxville
Bond
Memphis
J. F. Locke
MacQueen
Nashville
Clement
J. K. Peterson
Fisk University
Van Horn
George Peabody College for Teachers
F. L. Wren
Vanderbilt University
Miser
Sarratt
Sewanee
Bruton

TEXAS

Abilene
Abilene Christian College
Mullings
McMurray College
J. L. Tate
Simmons University
Burnam
Austin
Batchelder
H. Y. Benedict
H. V. Craig
Decherd
Dodd
Ettlinger
O. H. Hamilton
F. B. Jones
Lubben
R. L. Moore
G. H. Porter
M. B. Porter
Vandiver
Vickery
College Station
H. Halperin
Klipple
Dallas
P. V. Montgomery
Denton
Hanson
El Paso
Turrittin
Fort Worth
Sherer
Galveston
N. W. Wells
Houston
Ricker
Rust
Slotnick
Houston Junior College
W. A. Rees
Rice Institute
Beckenbach
Bray
Dean
L. R. Ford
Lovett
H. A. Wilson
Huntsville
Querry
Lubbock
Michie
E. L. Thompson
Marlin
Falconer
Mercedes
W. F. Shaw
San Antonio
Incarnate Word College
Mary de Lellis (Gough)
Our Lady of the Lake College
Mary Laetitia (Hill)
St. Mary's University of San Antonio
Schnepp
San Antonio Junior College
Hurry
Victoria
Tulloch
Waco
W. J. Williams
Waxahachie
G. A. Newton

UTAH

Logan
Linford
Tingey
Salt Lake City
Gibson
Hayes
Horsfall
Pehrson

VERMONT

Burlington
Bullard
Fraleigh
Swift
Northfield
L. E. Dix

VIRGINIA

Blacksburg
Hatcher
J. E. Williams
Charlottesville
Linfield
Luck
McShane
Oglesby
Sparrow
Dahlgren
Barker
L. T. E. Thompson
Fort Humphreys
Derby
Hampden-Sidney
E. R. Elliott
Hollins
M. J. Cox
Lexington
Byrne
Purdie
Lynchburg
Larew
Richmond
E. J. Willis
University of Richmond
Gaines
C. H. Wheeler
South Boston
Patten
Staunton
M. E. Taylor
Sweet Briar
N. Cole
Morenus
University
S. Buchanan
Pettis
G. T. Whyburn
Williamsburg
Stetson

WASHINGTON

Ellensburg
Trainor
Pullman
Biggerstaff
L. G. Butler
E. C. Colpitts
Isaacs
Seattle
Hostetter
Seattle Pacific College
Beegle
University of Washington
C. R. Ballantine
J. P. Ballantine
A. F. Carpenter
Cramlet
Gavett
Jerbert
McFarlan
Magnusson
Moritz
Mullemeister
Neikirk
Winger
Spokane
J. A. Carlson
Walla Walla
Bratton
Highberg
Yakima
C. Ross

WEST VIRGINIA

Bethany
Cramblet
Erskine
Huntington
Hackney
Institute
Claytor
Montgomery
W. F. Smith
Morgantown
Chambers
Colwell
H. A. Davis
R. H. Downing
Eiesland
C. N. Reynolds
B. M. Turner
Wheeling
Bagby

WISCONSIN

Beloit
Bisshopp
H. H. Conwell
Madison
Hayden
University of Wisconsin
G. E. Albert
F. E. Allen

T. Bennett
Breit
Erickson
H. P. Evans
I. G. Foster
V. C. Harris
W. W. Hart
Ingraham
Kiokemeister
Kleene
Langer
MacDuffee
March
May
Scheffé
Schwid
Slichter
E. S. Sokolnikoff
I. S. Sokolnikoff
Trump
E. B. Van Vleck
O. L. Wheeler
M. C. Wolf

Milwaukee
Bardell
P. H. Evans

Marquette University
Pettit
Wilczewski

Milwaukee-Downer College
Beckwith

State Teachers College
Knight
Norris

University of Wisconsin at Milwaukee
Battig
Marden
Nordhaus
Parkinson
W. E. Roth
Vass
L. A. Wolf

Oshkosh
Beenken
I. Price

River Falls
Eide

Superior
C. W. Smith

West De Pere
DeCleene

Williams Bay
Collins

Wisconsin Rapids
McMillan

WYOMING

Laramie
Bellamy

CANADA

Alberta

Edmonton
J. W. Campbell
A. J. Cook
Sheldon

British Columbia

Vancouver
D. Buchanan
Gage
Nowlan

Manitoba

Winnipeg
N. R. Wilson

New Brunswick

Sackville
W. H. McEwen

Nova Scotia

Halifax
Adshead

Wolfville
H. F. Archibald
Jeffery

Ontario

Hamilton
Findlay

Kingston
Gummer
Matheson
N. Miller
Watts

London
Kingston
Wiancko

Ottawa
G. V. Brady
A. W. Turner

Toronto
Wolfenden

University of Toronto
Beatty
Brauer
Burk
DeLury
Gilchrist
S. H. Gould
Krieger
Pounder
G. D. Robinson
Rosebrugh
Synge
Waddell
W. J. Webber

Quebec

Lennoxville
A. V. Richardson

Montreal
C. Craig
MacLean
Pall
A. N. Shaw
C. T. Sullivan
H. Tate
W. L. G. Williams

Quebec
Tremblay

Saskatchewan

Saskatoon
Ling

ARGENTINA

Buenos Aires
Baidaff

AUSTRIA

Vienna
Gödel

AUSTRALIA

Adelaide
Wilton

Perth
Weatherburn

Sydney
Thorne

CANAL ZONE

Quarry Heights
Sinkov

CHINA

Amoy
Chang

Canton
Woo

Peiping

National University of Peiping
Kiang

Tsing Hua University
Tseng

Tientsin
Chiang

CUBA

Havana
Corral-Alemán

DENMARK

Copenhagen

Royal Technical College
Jessen

University of Copenhagen
Bohr
Nörlund

FRANCE

Paris
Fréchet
Potron

GERMANY

Berlin
Bieberbach

Freiburg i. Br.
Bolza

Giessen
Geppert

Hamburg
Blaschke

Jena
König

Leipzig
Ackermann-Teubner
Deuring

GREAT BRITAIN

England

Birmingham
Watson

Bristol
Fraser

Cambridge
Macaulay

University of Cambridge
Carslaw
Collingwood
G. H. Hardy
Larmor
Ramsey
H. W. Richmond
Taussky

Leeds
W. P. Milne

London
Brill
Tavani
Western

University of London
Bosanquet
Hilton

Manchester
Mordell

Oxford
Dixon
E. B. Elliott
Frecheville
A. E. H. Love
Whitehead

Stocksfield-on-Tyne
Hardcastle

Suffolk
Curjel

Scotland
Dundee
Copson
Edinburgh
Jaramillo
Whittaker

Wales
Bangor
Berwick
Cardiff
Livens
Swansea
A. R. Richardson

GREECE
Athens
Zervos

HUNGARY
Budapest
Fejér
C. Jordan
Szeged
de Kerékjártó

INDIA
Aligarh
Shah
Annamalainagar
Rao
Bangalore
Iyengar
Bareilly
Bhattacharyya
Bombay
Dalal
Dacca
Vijayaraghavan
Delhi
Seth
Lucknow
Paranjpye
Madras
Ananda-Rau

IRAQ
Baghdad
Merrick

IRELAND
Dublin
McConnell
Rowe

ITALY
Palermo
Mineo
Pisa
Ricci
Tonelli
Rome
Levi-Civita
Volterra
Turin
Fano

JAPAN
Chibaken
Hirayama
Sapporo
Kawaguchi
Sendai
Kubota
Okada
Takasu
Tokyo
Kuniyeda
Yoshida
Yamagata
Yanagihara

NEW ZEALAND
Auckland
Forder

PALESTINE
Jerusalem
Levitzki

PANAMA
Panama City
Linares

PERU
Lima
de Losada y Puga

POLAND
Lwów
Ulam
Warsaw
Dickstein

SIAM
Bangkok
Hadlock

STRAITS SETTLEMENTS
Singapore
Oppenheim

SWEDEN
Stockholm
Oseen

SWITZERLAND
Zurich
Gut

TURKEY
Istanbul
Uzdilek

UNION OF SOVIET SOCIALIST REPUBLICS
Kieff
Kryloff
Leningrad
Smouroff

VENEZUELA
Caripito
C. H. Dix

BY-LAWS
OF THE
AMERICAN MATHEMATICAL SOCIETY

ARTICLE I

Officers

Section 1. The officers of the Society shall be a President, three Vice-Presidents, a Secretary, three Associate Secretaries, a Treasurer, a Librarian, and four Editorial Committees of three members each,—one for the *Bulletin*, one for the *Transactions*, one for the *Colloquium Publications*, and one consisting of the representatives of the Society on the Board of Editors of the *American Journal of Mathematics*.

Section 2. It shall be a duty of the President to deliver an address before the Society at the close of his term of office or within one year thereafter.

Section 3. The Librarian shall have charge of arrangements for the exchange of the Society's publications.

ARTICLE II

Board of Trustees

Section 1. There shall be a Board of Trustees consisting of five trustees elected by the Society, in accordance with Article IV.

Section 2. The function of the Board of Trustees shall be to receive and administer the funds of the Society, to have full legal control of its investments and properties, to make contracts and, in general, to conduct all business affairs of the Society.

ARTICLE III

Council

Section 1. The Council shall consist of the officers specified in Article I, of ex-secretaries who have served as Secretary for ten years or more, of ex-presidents for a period of six years after the expiration of their respective presidential terms, and of fifteen additional elected members chosen in accordance with Article IV, Section 2.

Section 2. Its function shall be to formulate and administer the scientific policies of the Society and to act in an advisory capacity to the Board of Trustees.

Section 3. At meetings of the Council five members shall constitute a quorum for the transaction of business. Between meetings business may be transacted by a mail vote. This shall require at least fifteen favorable votes and also that the number of these votes be three-quarters of those transmitted to the Secretary before the time announced for the closing of the polls. In case three or more members of the Council request postponement, at time of voting, action on the matter at issue shall be postponed until the next meeting of the Council, unless at the discretion of the Secretary the question is made the subject of a second letter vote, in connection with which brief statements of reasons, pro and con, are circulated to members.

ARTICLE IV

Election and Term of Office of Officers and Trustees

Section 1. The term of office shall be two years for the trustees, the President, the Vice-Presidents, the Secretary, the Associate Secretaries, and the Treasurer; three years each for the Librarian, the Editorial Committees of the *Bulletin*, the *Transactions*, and the *Colloquium Publications*, and the fifteen elected members of the Council, five of the latter retiring annually; five years for the representatives of the Society on the Board of Editors of the *American Journal of Mathematics*. In every case, however, the trustees and officers shall continue until their successors shall have been duly elected and qualified. The President, the Vice-Presidents, and the fifteen elected members of the Council shall not be eligible for immediate re-election to their respective offices.

Section 2. Election of trustees, officers, and members of the Council shall be

by ballot at the Annual Meeting. An official ballot shall be sent to each member by the Secretary at least one month prior to the Annual Meeting, and such ballots, if returned to the Secretary in envelopes bearing the names of the voters and received prior to the closing of the polls, shall be counted at the Annual Meeting. Each ballot shall contain a name proposed by the Council for each office to be filled, with blank spaces in which the voter may substitute other names. A majority of all votes cast, whether in person or by mail, shall be necessary for election. In case of failure to secure a majority for any office, the members present at the Annual Meeting shall choose by ballot between the two having the highest number of votes.

Section 3. If the President of the Society die or resign, before the expiration of his term of office, the Council may, with the approval of the trustees, designate one of the Vice-Presidents to serve as Acting President until the next Annual Meeting, when a President shall be elected by the Society. Such vacancies as may occur at any time among the other officers may be filled by the Council with the approval of the trustees.

Section 4. If any trustee die or resign during his tenure of office, the vacancy thus created shall be filled for his unexpired term by the Board of Trustees.

Section 5. If any elected member of the Council die or resign more than one year before the expiration of his term, the vacancy for the unexpired term shall be filled by the Society at the next Annual Meeting.

ARTICLE V

Election of Members

Section 1. Election of members shall be by vote of the Council.

Section 2. There shall be four classes of members, ordinary, contributing, institutional contributing, and sustaining.

Section 3. A firm, corporation, institution, association or individual interested in the support of mathematics may be elected to sustaining membership. A sustaining member shall receive both the *Bulletin* and the *Transactions* (see Article X), and shall have the privilege of nominating one or more persons for election by the Council to ordinary membership in the Society, the number to be determined by the Council. Such nominations shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail).

Section 4. A firm, corporation, institution, or association interested in the support of mathematics may be elected to institutional contributing membership. A contributing member shall receive the *Bulletin* or have the privilege, for each twenty-five dollars contributed, of nominating one person to ordinary membership in the Society. Such nominations shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail).

Section 5. Except in the case of nominees of sustaining members and of institutional contributing members, application for admission to ordinary membership shall be made by the applicant, on a blank provided by the Secretary, and shall be approved by two members of the Society. Such applications shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail), except in the case of members of other societies entering under special action of the Council.

Section 6. An ordinary member may become a contributing member by paying the dues for such membership.

ARTICLE VI

Dues

Section 1. Persons elected to ordinary membership in the Society by the Council, under the provisions of Article V, Section 5, shall be admitted to membership upon the payment, within sixty days of the date of their election, of an initiation fee of five dollars.

Section 2. The annual dues of persons elected by the Council to ordinary membership under the provisions of Article V, Section 5, shall be eight dollars, with the following exceptions: (1) during the first three years of membership, the annual dues shall be six dollars; a person shall be considered to have completed his first year of membership on January 1 following his election; (2) the amount of dues may be altered by reciprocity agreements with other societies;

(3) the Council may make special rulings in exceptional cases, with the approval of the Board of Trustees; four dollars and fifty cents of the dues of each member shall be for a year's subscription to the *Bulletin*. Each new member shall pay in proportion to the unexpired fraction of the year at the time of his election.

Section 3. The minimum dues for a contributing member shall be fifteen dollars per year. Members may, upon their own initiative, pay larger dues.

Section 4. The dues for an institutional contributing member for any year shall not be less than twenty-five dollars. Institutions may pay larger dues.

Section 5. The dues of a sustaining member for any year shall not be less than one hundred dollars. A sustaining member who contributes annually at least five hundred dollars shall be designated as a Patron of the Society.

Section 6. Persons elected to ordinary membership as nominees of sustaining members or institutional contributing members under the provisions of Article V, Sections 3-4, shall not be required to pay an initiation fee. They shall not be required to pay dues, so long as it is agreed that they are designated by the sustaining member or institutional contributing member under these provisions. If a nominee of a sustaining member or an institutional contributing member later becomes a dues-paying member, he shall pay dues at the rate of six dollars a year for the remainder (if any) of his first three years of membership, and eight dollars a year thereafter.

Section 7. If the annual dues of any member remain unpaid beyond a reasonable time, the Board of Trustees shall remove his name from the list of members, after due notice.

Section 8. Any member not in arrears of dues may become a life member on the payment of a sum to be determined in accordance with actuarial principles.*

ARTICLE VII

Meetings

Section 1. The Annual Meeting of the Society shall be held between the fifteenth of December and the fifteenth of January next following. Notice of the time and place of this meeting shall be mailed by the Secretary or an Associate Secretary to the last known post office address of each member of the Society. The times and places of the Annual and other meetings of the Society shall be designated by the Council.

Section 2. The President may call a meeting of the Council whenever the affairs of the Society make it desirable; he shall call a meeting of the Council on written request of five of its members.

Section 3. The Board of Trustees shall have its regular Annual Meeting on the first day of the Annual Meeting of the Society. Special meetings of the Board of Trustees may be called by the Chairman of the Board upon three days' notice of such meeting mailed to the last known post office address of each trustee. He shall call a meeting upon the receipt of a written request of two of the trustees. Meetings for the transaction of business may also be held by common consent of all the trustees.

ARTICLE VIII

Sections

Whenever it shall appear to the Council that a sufficient number of members of the Society are desirous of conducting in any locality periodic meetings for the reading and discussion of mathematical papers, the Council may authorize the formation of a Section to be composed at each sectional meeting of such members of the Society as may be present; and the Council shall have the right to withdraw such authorization.

ARTICLE IX

Programs

Section 1. Papers intended for presentation at any meeting or sectional meeting of the Society shall be passed upon in advance by a program committee appointed by or under the authority of the Council; and only such papers shall be presented as shall have been approved by such committee. Papers in form

* See tab'e on page 80.

unsuitable for publication, if accepted for presentation, shall be referred to on the program as preliminary communications or reports.

Section 2. No matter of general business shall be considered at any meeting of the Society except the Annual Meeting without the recommendation of the Council.

ARTICLE X

Publications

Section 1. The Society shall publish an official organ called the *Bulletin of the American Mathematical Society* which shall be sent to each member of the Society; four dollars and fifty cents of the dues paid by each member shall be for a year's subscription to this publication. It shall publish a series of volumes called *Colloquium Publications*, which shall embody in book form new mathematical developments. It shall publish a journal, the object of which shall be to make known important researches presented at meetings of the Society; this journal shall be called the *Transactions of the American Mathematical Society*. It shall also cooperate in the conduct of the *American Journal of Mathematics*.

Section 2. The editorial management of the *Bulletin*, *Transactions*, and *Colloquium Publications* and the participation of the Society in the editorial management of the *American Journal of Mathematics* shall be in charge of committees as specified in Article I.

ARTICLE XI

Amendments

These By-Laws may be amended or suspended at any meeting of the Society on recommendation of the Council and by a two-thirds vote of the members present, provided notice of such proposed action and of its general nature shall have been given in the call for such meeting.

REPORT OF THE TREASURER FOR THE YEAR 1934

Bulletin Fund

Receipts:		
Balance from 1933 (including reprinting fund ($1,267.22) and reserve for List of Members ($400.00))		$ 4,610.84
Appropriation from Dues for current volume		6,500.00
Appropriation from Sustaining Memberships		100.00
Appropriation from Special Contributions		125.00
Rockefeller Foundation Subvention (1934)		500.00
Sales, etc.:		
Reprints	$ 102.51	
Back volumes	324.08	
Subscriptions	1,651.34	2,077.93
		$13,913.77
Expenditures:		
Printing Bulletin (3 numbers for 1933, 10 numbers for 1934)	$6,264.23	
Printing List of Members	364.72	
Repaid to General Funds for paper for 1934	950.40	
Mailing, clerical, insurance, storage (including ½ clerk's salary, and $392.49 for work on List of Members)	3,192.70	
Clerical work on 10-year index	15.25	
Editors' travelling	206.45	10,993.75
Balance (including $1,576.05 reprinting and index fund)		$ 2,920.02

Transactions Fund

Receipts:		
Balance from 1933 (including $365.82 reprinting fund)		$ 411.25
Rockefeller Foundation Subvention (1934)		1,500.00
Appropriation from Sustaining Memberships		400.00
Appropriation from Special Contributions		895.64
Endowment Fund Interest		2,000.00
Sales, etc.:		
Reprints	$ 206.04	
Back volumes	1,093.32	
Subscriptions	3,347.94	4,647.30
		$ 9,854.19
Expenditures:		
Printing (4 numbers for 1934)	$5,027.52	
Repaid to General Funds for paper for 1934	589.14	
Mailing, clerical, insurance, storage (with ½ clerk's salary)	2,166.31	7,782.97
Balance (including $459.14 reprinting fund)		$ 2,071.22

Colloquium Fund

Receipts:		
Balance from 1933		$ 6,331.24
Rockefeller Foundation Subvention (1934)		1,000.00
Sales (not including Bliss, Algebraic Functions)*		2,297.26
Colloquium Attendance Fees		184.00
		$9,812.50
Expenditures:		
Reprinting Princeton Lectures	$ 490.00	
Reprinting Eisenhart	500.00	
Wedderburn	1,523.02	

* (Bliss, Algebraic Functions: total receipts $551.80; charge made by Society for shipping, $165.00; returned to National Research Council, $386.80.)

Morse	2,618.53	
Shipping, insurance, storage (not including Bliss)	772.53	
Circulars and advertisements	98.04	
Travelling and honorariums	110.00	6,112.12
Balance		$ 3,700.38

GENERAL ACTIVITIES

Balance at end of 1933 (including $1,500.00 Sinking Fund and not including advances for paper)		$ 2,683.67
Correction because of increase in Life Membership Reserve ($6,883.92–$6,588.41)		295.51
Advances for paper returned by Bulletin and Transactions Funds		1,539.54
		$ 4,518.72
Receipts:		
Annual dues	$11,740.84	
Initiation fees	251.00	
Life Membership adjustment	22.44	
Interest		
from Endowment Fund	705.33	
from General Securities	572.28	
from Bank Interest	299.64	
Sustaining Memberships	865.00	14,456.53
		$18,975.25
Expenditures:		
Clerk's annuity	$ 265.05	
Secretaries	3,529.30	
Treasurer	900.00	
Officers' travelling	461.42	
Committee on Financial Policy	200.00	
Euler Subvention	100.00	
Emergency Fund	22.64	
Furniture and Moving	333.83	
	$ 5,812.24	
Bulletin appropriation for 1934	6,500.00	
Library appropriation	400.00	12,712.24
Balance (including $1,500.00 Sinking Fund)		$ 6,263.01

LIBRARY FUND

Receipts:	
Balance from 1933	$ 21.32
Appropriation from Dues	400.00
Sale of duplicates	18.90
	$ 440.22
Expenditures:	
Books, binding, clerical	440.22
Balance	00.00

STABILIZATION FUND

Receipts (Rockefeller Subvention)	$ 4,250.00
Expenditures	2,615.18
Balance	$ 1,634.82

American Journal Fund

Receipts:		
Rockefeller Foundation Subvention (1934)		$1,500.00
Endowment Fund Interest		250.00
		$1,750.00
Expenditures (Subvention)		1,750.00
Balance		00.00

Summary

Life Membership Reserve, 1933		$ 6,588.41
Balance, General Activities Fund		6,263.01
Balance, Bulletin Fund		2,920.02
Balance, Transactions Fund		2,071.22
Balance, Colloquium Fund		3,700.38
Balance, Stabilization Fund		1,634.82
		$23,177.86
Investments (1933)	$8,125.27	
Securities purchased April 15	5,548.61	13,673.88
		$ 9,503.98

The amount of reserve necessary to protect the life memberships is now $6,247.62.

As reported by Mr. Roosevelt, the par value of the securities held by the Society is now $98,500.00.

Endowment Fund, Sustaining Memberships, Special Contributions

Receipts:		
Cash balance, 1933		$ 522.61
From pledges and gifts		68.00
From Sustaining Memberships		1,365.00
From Special Contributions		1,020.64
		$ 2,976.25
Expenditures:		
From Sustaining Memberships to General Activities	$ 865.00	
From Sustaining Memberships to Bulletin	100.00	
From Sustaining Memberships to Transactions	400.00	
From Special Contributions to Bulletin	125.00	
From Special Contributions to Transactions	895.64	2,385.64
		$ 590.61

The Endowment Fund has now securities of value $72,568.26 (as amortized March, 1934) (including $100 Ettlinger Fund). The interest in 1934 (less charges for handling of Fund) was $2,955.33.

Special Funds

Bôcher Fund:		
Balance from 1933:		
Investments	$1,163.60	
Cash	742.25	
		$ 1,905.85
Receipts (interest)		108.42
		$ 2,014.27

Expenditures (prizes)		200.00
Balance		$ 1,814.27
Principal	$1,188.00	
Interest	626.27	
		$ 1,814.27

Cole Fund:

Balance from 1933:		
Investments	$2,023.02	
Cash	684.66	
		$ 2,707.68
Receipts (interest)		144.67
Balance		$ 2,852.35
Principal	$2,093.13	
Interest	759.22	
		$ 2,852.35

Moore Fund:

Balance from 1933:		
Investments	$2,973.81	
Cash	648.32	
		$ 3,622.13
Receipts (interest)		164.93
Balance		$ 3,787.06
Principal	$2,100.62	
Interest	1,686.44	
		$ 3,787.06

G. W. MULLINS,
Treasurer.

NEW YORK, N.Y., December 1, 1934.

REPORT OF THE AUDITING COMMITTEE, 1934

We, the undersigned Auditing Committee, appointed by the American Mathematical Society, have this day audited the accounts of the Treasurer and in our opinion the following statement is correct:

GENERAL FUNDS

Balance from 1933	$20,942.24
Cash receipts	36,367.56
	$57,309.80
Disbursements	34,131.94
Balance, December 1, 1934	$23,177.86

Invested as follows:

Cash in Corn Exchange Bank, exclusive of outstanding checks	$ 3,331.61
Cash in Union Dime Savings Bank	3,616.14
Cash in Bank for Savings	2,556.23
Investments	13,673.88
	$23,177.86

SPECIAL FUNDS

Bôcher Fund:

Balance from 1933		$ 1,905.85
Receipts (interest)		108.42
		$ 2,014.27
Expenditures		200.00
Balance		$ 1,814.27
Investments	$1,663.60	
Cash	150.67	
		$ 1,814.27

Cole Fund:

Balance from 1933		$ 2,707.68
Receipts (interest)		144.67
Balance		$ 2,852.35
Investments	$2,523.02	
Cash	329.33	
		$ 2,852.35

Moore Fund:

Balance from 1933		$ 3,622.13
Receipts (interest)		164.93
Balance		$ 3,787.06
Investments	$2,973.81	
Cash	813.25	
		$ 3,787.06

The amount of reserve necessary to protect the life membership is now $6,247.62.

B. P. GILL
H. W. REDDICK
J. J. TANZOLA
Auditing Committee

NEW YORK, N.Y., December 12, 1934.

REPORT OF THE TREASURER FOR THE YEAR 1935

BULLETIN FUND

Receipts:

Balance from 1934 (including reprinting fund $1,576.05)		$ 2,920.02
Appropriation from Dues for current volume		8,400.00
Sales, etc.:		
Reprints	$ 76.03	
Back volumes	170.76	
Subscriptions	1,618.88	1,865.67
		$13,185.69

Expenditures:

Printing Bulletin (3 numbers for 1934, 9 numbers for 1935)	$6,155.50	
For Treasurer's expenses	344.51	
Clerk's annuity	130.05	
Mailing, clerical, insurance, storage, etc.	2,488.90	
Clerical work on 10-year index	434.33	9,553.29
Balance (including reprinting fund, $1,312.48 and reserve for List of Members, $400)		$ 3,632.40

Transactions Fund

Receipts:		
Balance from 1934 (including reprinting fund $459.14)		$ 2,071.22
Rockefeller Foundation Subvention (1935)		2,000.00
Appropriation from Institutional Memberships		550.00
Appropriation from Individual Contributing Memberships		453.50
Appropriation from Special Contributions		64.50
Endowment Fund Interest		1,500.00
Sales, etc.:		
Reprints	$ 133.01	
Back volumes	506.44	
Subscriptions	4,628.93	5,268.38
		$11,907.60
Expenditures:		
Printing (6 numbers for 1935)	$6,648.48	
Mailing, clerical, insurance, storage, etc.	2,120.67	8,769.15
Balance (including reprinting fund, $459.14)		$ 3,138.45

Colloquium Fund

Receipts:		
Balance from 1934		$ 3,700.38
Rockefeller Foundation Subvention (1935)		1,000.00
Sales (not including Bliss, Algebraic Functions)*		4,486.19
Colloquium Attendance Fees		192.00
Special Contributions		100.00
		$ 9,478.57
Expenditures:		
Printing Paley & Wiener	$1,480.28	
Customs and duty, importing Jackson and Lefschetz	95.84	
Shipping, insurance, storage, clerical	912.00	
Circulars and advertisements	70.88	
Travelling and honorariums	134.46	2,693.46
Balance		$ 6,785.11

General Activities

Balance at end of 1934 (including $1,500 Sinking Fund)		$ 6,263.01
Correction resulting from change in Life Membership Reserve ($6,588.41–$6,247.62)		340.79
		$ 6,603.80
Receipts:		
Annual dues	$11,724.02	
Initiation fees	345.00	
Life Membership	124.64	
Interest		
from Endowment Fund	963.03	
from General Securities	554.76	
from Savings Accounts	186.95	
Institutional Memberships	5,055.00	
Other contributions toward defrayment of publication cost	644.88	
Individual Contributing Memberships	453.50	
Special Contributions	34.00	20,085.78
		$26,689.58

* (Bliss, Algebraic Functions: total receipts, $366.11; charge made by Society for shipping, $53.00; returned to National Research Council, $313.11.)

Expenditures:		
Clerk's annuity	$ 135.00	
Secretaries	4,400.00	
Treasurer	550.00	
Officers' travelling	358.51	
Membership Committee	200.00	
Euler Subvention	100.00	
Emergency Fund	451.53	
Office furniture	187.66	
Gibbs Lecture	69.74	
	$ 6,452.44	
Bulletin appropriation for 1935	8,400.00	
Library appropriation	500.00	15,352.44
Balance (including $1,500 Sinking Fund)		$11,337.14

LIBRARY FUND

Receipts:		
Balance from 1934		00.00
Appropriation from Dues		$ 500.00
Sale of duplicates		64.00
		$ 564.00
Expenditures:		
Binding, clerical, etc.		314.32
Balance		$ 249.68

STABILIZATION FUND

Receipts:	
Balance from 1934	$ 1,634.82
Rockefeller Subvention	4,250.00
	$ 5,884.82
Expenditures:	
Salary, travelling expenses, etc.	5,823.47
Balance	$ 61.35

AMERICAN JOURNAL FUND

Receipts:	
Rockefeller Foundation Subvention (1935)	$ 1,500.00
Endowment Fund Interest	400.00
	$ 1,900.00
Expenditures:	
Subvention to American Journal	1,900.00
Balance	00.00

SUMMARY

Life Membership Reserve, 1934	$ 6,247.62
Balance, General Fund	11,337.14
Balance, Bulletin Fund	3,632.40
Balance, Transactions Fund	3,138.45
Balance, Colloquium Fund	6,785.11
Balance, Stabilization Fund	61.35
Balance, Library Fund	249.68
	$31,451.75
Investments	13,673.88
Cash Balance	$17,777.87

The amount of reserve necessary to protect the life memberships is now $6,098.80.

As reported by Mr. Roosevelt, the par value of the securities held by the Society is now $98,500. There are now securities of par value $24,000 which are defaulting interest payments.

ENDOWMENT FUND, SUSTAINING MEMBERSHIPS, SPECIAL CONTRIBUTIONS

Receipts:		
Cash balance, 1934 (Endowment Fund)		$ 590.61
Pledges and gifts to Endowment Fund		25.00
Institutional Memberships		5,605.00
Contributions for publication expense		644.88
Individual Contributing Memberships		907.00
Special Contributions		198.50
		$ 7,970.99
Expenditures:		
From Institutional Memberships to General Activities	$5,055.00	
From Institutional Memberships to Transactions	550.00	
From contributions for publication expense to General Activities	644.88	
From Special Contributions to General Activities	34.00	
From Special Contributions to Transactions	64.50	
From Special Contributions to Colloquium	100.00	
From Contributing Memberships to Transactions	453.50	
From Contributing Memberships to General Activities	453.50	7,355.38
Balance		$ 615.61

The Endowment Fund has now securities of value $72,568.26 (as amortized March, 1934) (including $100 Ettlinger Fund). The interest in 1935 (less charges for handling of Fund) was $2,863.03.

SPECIAL FUNDS

Bôcher Fund:		
Balance from 1934:		
Investments	$1,663.60	
Cash	150.67	
		$ 1,814.27
Receipts (interest, 1935)		78.13
Balance		$ 1,892.40
Principal	$1,188.00	
Interest	704.40	
		$ 1,892.40
Cole Fund:		
Balance from 1934:		
Investments	$2,523.02	
Cash	329.33	
		$ 2,852.35
Receipts (interest, 1935)		119.28
Balance		$ 2,971.63
Principal	$2,093.13	
Interest	878.50	
		$ 2,971.63

Moore Fund:

Balance from 1934		
Investments	$2,973.81	
Cash	813.25	
		$ 3,787.06
Receipts (interest, 1935)		131.77
Balance		$ 3,918.83
Principal	$2,100.62	
Interest	1,818.21	
		$3,918.83

G. W. MULLINS,
Treasurer.

NEW YORK, N.Y., December 2, 1935.

REPORT OF THE AUDITING COMMITTEE, 1935

We, the undersigned Auditing Committee, appointed by the American Mathematical Society, have this day audited the accounts of the Treasurer and in our opinion the following statement is correct:

GENERAL FUNDS

Balance from 1934	$23,177.86
Cash receipts	43,780.02
	$66,957.88
Disbursements	35,506.13
Balance, December 2, 1935	$31,451.75
Invested as follows:	
Cash in Corn Exchange Bank, exclusive of outstanding checks	$ 6,724.15
Cash in Union Dime Savings Bank	3,400.72
Cash in Bank for Savings	2,620.71
Cash in Providence Bank for Savings	5,032.29
Investments	13,673.88
	$31,451.75

SPECIAL FUNDS

Bôcher Fund:

Balance from 1934		$ 1,814.27
Receipts (interest)		78.13
Balance		$ 1,892.40
Investments	$1,663.60	
Cash	228.80	
		$ 1,892.40

Cole Fund:

Balance from 1934		$ 2,852.35
Receipts (interest)		119.28
Balance		$ 2,971.63
Investments	$2,523.02	
Cash	448.61	
		$ 2,971.63

HE TABLE OF LIFE MEMBERSHIP FEES

ars may be obtained from the Treasurer.)

Age	Fee
55	$94.36
60	82.74
65	70.71
70	58.81
75	47.56
80	37.44

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Moore Fund:

Balance from 1934	
Receipts (interest)	
Balance	
Investments	
Cash	

The amount of reserve necessary to protec
$6,098.80.

NEW YORK, N.Y., December 12, 1935.

being derived from another source. The most convenient way is by a torque amplifier [16], which is a device whereby one shaft moves another precisely in step, but with much larger torque delivered than supplied. This works on the same principle as the capstan or windlass, and depends upon wrapped bands. The input shaft pulls one end of the band, and the band rubs on a continuously revolving drum and pulls the output shaft along. Two drums provide for both directions of rotation. Units are in continuous operation giving multiplying factors as high as 10,000.

After simple integration the next important operation is the integration of a function multiplied by a standard kernel. The most prevalent form of device for performing such an operation is the harmonic analyser, for evaluating the integral of the product of a function by a sinusoid. Great ingenuity has been shown in devising such instruments, and scores have been invented. It is not much exaggerated to state that as many forms have been invented as there are actual instruments in present use. Perhaps this is not undesirable, for it is certainly much more pleasant to invent a device of this nature than it is to operate the finished product. The writer pleads guilty to having invented several, none of which are in use. Most harmonic analysers are mechanical combinations of variable speed drives used as integrators, with link motions or cams for introducing the sinusoidal factor. Such, for example, is the Chubb machine [17]. The most convenient and precise is the Henrici-Coradi [18], which introduces its sinusoidal components by means of rollers operating on the surface of glass spheres. There are, however, optical devices, those depending on spring deflections, devices for displacing fluids, and many others. An harmonic analyser, when inverted, becomes a synthesizer. Such are, for example, the tide predicting machines, of which the first was Kelvin's [19].

Of greater mathematical scope, however, is a device for obtaining instantaneously the integral of the product of two given functions. Suggested by Wiener, it has been developed over several years by Gould, Gray, and Hazen [20]. The functions are represented by optical masks placed in parallel planes. The light from a linear light source which passes through both masks is summed by a photo-cell, and gives the measure of the integral

of the product. If one of the masks is now shifted in the direction of abscissas, the effect is to shift the origin of one of the functions. The value of the light received plotted against the shift thus evaluates an integral with a cyclic kernel as a function of the parameter under the sign of integration. Work is now under way which aims, in effect, at deforming one of the masks in predetermined fashion, thus evaluating an integral with a general kernel. Such an instrument, which has been called a cinema integraph, is immediately applicable for the evaluation of Fourier transforms, for correlation analysis, and many other purposes.

There is a very fundamental difference between this last instrument and the usual integrator, a difference of profound mathematical importance. Evaluating an integral of a product of functions is one thing, but examining the variation of the integral when one of the functions is deformed or shifted is quite another. There are many devices for performing operations. The most interesting effects occur, however, when these are combined.

The more important class of instruments is therefore the second, the equation solvers. There are three principal types. In the first, simple analogy is used. To solve the equations controlling the performance of a given system, a second system is set up which obeys the same laws, and its performance is measured. The benefit of the substitution is merely one of convenience and precision in measurement. Thus, it is relatively easy to measure the flow of a fluid and hard to measure the flow of heat. Hence, if one wishes to examine the transient flow of heat in a complex device, where analysis is out of the question, such, for example, as a complicated system of hot pipes imbedded in an insulator, one may construct an analogous system where the flow of heat is replaced by the flow of a viscous fluid and measure that. All that is necessary is to provide an apparatus representing the cross section of the system by two plates close together with holes representing the pipes, and supply fluid to each of these holes at pressures proportional to the corresponding pipe temperature. Inserting dye at spaced points will give the flow lines, as was done by Hele-Shaw and his collaborators [21]. Collecting and measuring the fluid gives the total heat flow directly.

A variation of this scheme is to employ a system of the same

general type, but on a different and usually reduced scale. The dimensions and parameters are so chosen that the original equations still hold, with modification only by multiplication of variables by constants. The systems are, in other words, arranged to be dynamically similar. This is the scheme of the wind tunnel [22]. A small airplane, or a small ship, is tested and the results then interpreted to apply to the full-sized device. In each case the velocity through the fluid is usually reduced, as well as the dimensions. There is a complication in this process. The resistance to the passage of a solid through a fluid is of at least two sorts, and they obey different laws. Thus a ship has wave resistance and skin resistance [23]. The same scheme of scaling down will not apply to both. Hence it is necessary to render one negligible or measure it separately, and great difficulty in attaining high precision ensues, which is the reason that the use of towing tanks and wind tunnels causes so many arguments, and the reason why large ones are preferred.

In the second type of equation-solver, the constants of the system being studied are independently represented, and this leads to flexibility. An electrical example of this second type of instrument for the solution of equations is given by the Network Analyser [24]. It is primarily intended for the examination of complex electrical power networks, and it hence solves certain restricted classes of simultaneous algebraic equations with complex coefficients. It consists merely of a set of condensers, coils, and resistances which may be connected together by plugs. There is thus produced an exact small-scale electrical replica of the system being studied; a miniature network having the same electrical proportions as the actual system. To this are applied, at various points, voltages proportional to those applied to the actual network, and in the same phase relations. The voltages, and hence the currents, are scaled down, and hence the model may be physically small. Measurements made with ordinary instruments yield the performance of the full-size system under interesting operating or emergency conditions. A power system covering several states is thus compacted into a single room for detailed examination. This type of instrument forms an intermediate step between the simple model and the completely flexible equation-solver, for the Network Analyser may be reconnected to represent any desired power system within its

limits, and thus one instrument serves for the solution of problems concerning many power networks.

The third type of equation-solver is, however, the one with which we are now most concerned. Like the others, it provides a new system, which obeys the same laws as the one under investigation, and which may be readily and precisely measured, but it does so in a different way. Instead of carrying over an entire equation from a system to its substitute, or even the constants of the equation as in the Network Analyser, the terms of the equation are carried over individually as such and then recombined. This gives great flexibility, for, since the mode of combination is under control, many diverse systems may be studied by a single substitute instrument. As soon as elements are available for performing the operations indicated in an equation, whatever they may be, one thing only is necessary to make an instrument for solving the equation. That is to close the train of operations, or "back-couple" the device. If the equation admits of a discrete solution, this process fixes the parts in position. If the solution is a relation between variables, the device is constrained to move only in accordance with that relationship.

The instrument for solving simultaneous linear algebraic equations gives an example. Wilbur has produced a mechanical device for this purpose, modifying a suggestion which originated, as did so many ingenious ideas, with Kelvin [25]. Unknowns are represented by the angles of platens. The positions of a set of pulleys represent the coefficients. Steel bands running over these pulleys have an effective lengthening or shortening representing the terms of the equations, each band corresponding to one equation. Each band passes over a pulley on each platen, thus summing the terms of the corresponding equation. On setting the ends of the bands on indices representing the constant terms, the entire device is fixed in position, and the angles may be read and hence the unknowns determined. The device is entirely geometrical, and no elastic deformation of parts is employed. Before the last band is anchored to its index, all platens are free to move. Fixing this last band completes the set-up and locks every element in position, provided the equations are determinate.

Mallock has built an electrical device for this same purpose [26]. The variables are represented by alternating currents in

closed circuits, which are coupled together by transformers with adjustable numbers of turns, made proportional to the coefficients, there being as many coils on the transformer as there are variables, plus one to represent the constant terms. Since the total ampere-turns on a transformer must be zero, except for a small excitation, the equity is thus forced. Mallock has a very ingenious way of supplying the excitation separately, to avoid error.

Higher order algebraic equations may be handled in various ways, but so far not precisely [27]. The use of the displacement of solids dipped into liquids, as exemplified by a proposal by Meslin, gives one method of attack.

The third type of equation-solver is best exemplified by the differential analyser [15], which is an instrument for solving ordinary differential equations. It consists essentially of a set of integrators and means for interconnecting them. Kelvin made the suggestion of this procedure [28], but much development was necessary to reduce the idea to practice. There is no limit to the complexity of the equations which may be treated except the number of units available. Several differential-analysers have now been built, and are being used on a large number of problems. Detailed descriptions are available, so only a few points will be mentioned.

The manner of "back-coupling" the device is of especial interest. A single integrator is merely an instrument for performing the operation of integration. If its disc be turned at constant speed, having at any time an angle x, and if its displacement be varied in accordance with a variable y, its output will yield the value of $\int y\,dx$. Suppose now, however, that its output be connected to the y shaft. This is back-coupling. It forces y at all instants to be equal to the output of $\int y\,dx$. The machine is now constrained, and can move only in one definite way, namely, in accordance with the equation $y = \int y\,dx$ or $dy/dx = y$.

It thus yields the exponential solution of this equation, if simultaneous readings be taken of the positions of the x and y shafts. This is the simplest possible sort of back-coupling. Actually, in interesting problems, several integrators are interconnected, together with differential gears for adding, and output tables for introducing variable coefficients, before the final connection is made which fixes the performance of the machine.

The way in which an equation is placed upon the machine may be traced. It is first solved formally for the highest order derivative, and integrated once formally, to yield the derivative of next lower order. A shaft is assigned to this, and one to the independent variable. An integrator may now be connected to yield the derivative of next lower order, and so on until the dependent variable is reached. As we proceed in this manner, every term in the equation becomes represented by the revolutions of a shaft. These shafts are then interconnected through differential gears so that their revolutions sum to zero. This closes the equation, and in effect provides for the drive of the shaft assigned to the next to highest order derivative. When the independent variable shaft is now turned, every other shaft is driven, and the machine is constrained to move in accordance with the equation. The initial conditions are introduced by setting the starting portions of the integrators.

Two terms may be multiplied by a separate mechanism, or better by summing their cross integrals. The square of a variable may be obtained by integrating it against itself. Functional coefficients may be introduced manually as the solution proceeds; but it is more convenient, and also more accurate, to generate them wherever possible. Any coefficient which can be obtained as the solution of an ordinary differential equation may be thus generated, by assigning a portion of the machine to this duty. When thus treated an ordinary differential equation with variable coefficients becomes a more complicated set of equations, but with constant coefficients.

Manual introduction of functions has however one interesting consequence. The procedure is as follows: An index lies on a plotted curve. It is driven in the direction of abscissas by any desired variable by coupling to the machine at the proper point. It is cranked in the direction of ordinates by hand so as to remain on the curve. The crank also turns the shaft in the machine which represents the function. But this makes possible another procedure. The curve being followed may be the actual record being traced by the machine as a solution, the abscissal motion of the index being controlled by a separate variable, a fraction of the variable of abscissas of the curve, or some other. A wide range of functional equations may thus be handled.

It is interesting to speculate on the effect of coupling the

cinema integraph to the differential analyser. One of the terms of the equation being solved may then involve a variable parameter under the sign of integration, and the other terms may be those of an ordinary or functional differential equation.

This does not, of course, treat the type of integral equation which is equivalent to a partial differential equation plus a set of boundary conditions. The cinema integraph is adapted to do this by successive approximations, but not directly. The differential analyser also has possibilities in this regard, for it can handle the equations of the characteristics and thus make the solution indirectly. The direct approach would require the continuous deformation of one of the masks in such manner as to cause equality of the output to a specified combination. This has not yet been attempted.

This brings us to the final section of this paper. What may be the mutual influence in the future between machines of the sort considered and formal mathematics?

The machine will not, of course, yield a formal result; it will give only approximate solutions. But its limitations are not at all those usually attending formal solutions. The machine does not care how complex the expression for a coefficient may be, so long as it may be plotted. Discontinuities bother it not at all. Bizarre combinations, such as a function of a derivative appearing as a coefficient, are exactly as readily provided for as is the usual case.

There is no use thus solving an equation unless its solution means something concerning a physical system or unless it suggests the form of a formal solution. In the past we have treated physical problems by means of equations that, allegedly at least, could be solved formally, and the equations have been rather simple in their make-up. Is this because nature prefers the sort of equation that we can handle formally? Or has there been a sifting process whereby the simple equations only have survived?

The equations that come to the differential analyser are "reasonable" equations, in that they admit of solution in a reasonable time of operation of the machine; whereas it is readily possible to construct arbitrarily equations which are most unreasonable in this respect. Why is nature so reasonable, or is the reasonableness ours?

Is there any real use or meaning to some of the bizarre equations that can be solved? Certainly the functional equation crops up, although it is sparingly used in physical problems.

Entirely aside from this speculation, there is one influence which is undoubtedly coming, if equation-solvers become as fully developed and as rapid, reliable, and versatile as the arithmetical machines of commerce. Formal attention will be less directed to the mere solution of equations, in order that they may thus be rendered useful; and will hence be more directed to their formulation and interpretation.

It is to be hoped, as well, that formal attention will be directed to the machines themselves. There are many fascinating problems involved, many of them much too mathematical for those who are engaged in the detailed machine developments. An important query is the question of what sort of machines are really worth developing. This involves the mathematician fully as much as the user of the results.

My mathematical friends exclaim over the ingenuity of the formalist in inventing new methods of construction for use in existence proofs. Perhaps photo-cells might be of service.

We are in an age of complex instruments. Out of it will come devices that will revolutionize the use of mathematics, and will profoundly influence some branches of mathematics itself. This process is now beginning, and it is probable that the next decade will see important advances.

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A CERTAIN MEAN-VALUE PROBLEM IN STATISTICS*

BY A. T. CRAIG

1. *Introduction.* It is the purpose of this paper to investigate, by means of the characteristic function, the arithmetic mean value, or mathematical expectation, of the sum of the squares of n normally and independently distributed variables when those variables are subject to $m<n$ linear restrictions. For example, if $x_1, x_2, \cdots, x_n$ are n independent values of a variable x which is normally distributed with mean zero and variance σ^2, then the expected value of $\sum_1^n x_j^2$ is $n\sigma^2$. However, the expected value of $\sum_1^n (x_j-\bar{x})^2$, where $n\bar{x}=\sum_1^n x_j$, is $(n-1)\sigma^2$. It is fairly obvious that the latter example could be stated: if the x's are subject to the linear restriction $\sum_1^n x_j=0$, the expected value of $\sum_1^n x_j^2$ is $(n-1)\sigma^2$. The numbers n and $n-1$, which are equal respectively to the ranks of the matrices of the two quadratic forms, are frequently called the number of degrees of freedom of those quadratic forms.

Let x be subject to the normal law of error

$$f(x) = \frac{1}{\sigma(2\pi)^{1/2}} e^{-x^2/2\sigma^2}$$

and let $x_1, x_2, \cdots, x_n$, be n independent values of x. Write

$$v = \sum_1^n x_j^2, \qquad u_1 = \sum_1^n a_{1j}x_j, \cdots, \qquad u_m = \sum_1^n a_{mj}x_j,$$

in which the a's are real numbers. We wish to find the mathematical expectation of v when $u_1, u_2, \cdots, u_m$ are assigned values which make the system consistent. It is well known that the variables $u_1, u_2, \cdots, u_m$ are normally correlated with variances and covariances given by $\sigma^2\sum_r a_{jr}a_{kr}$.

2. *The Characteristic Function.* The characteristic function of the joint distribution of $v, u_1, \cdots, u_m$ is

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* Presented to the Society, April 11, 1936.

$$\phi(t_1, t_2, \cdots, t_m, t_{m+1}) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \int \cdots \int e^{\theta} dx_n \cdots dx_1,$$

where

$$\theta = it_1 \sum_1^n a_{1j}x_j + \cdots + it_m \sum_1^n a_{mj}x_j + \left(it_{m+1} - \frac{1}{2\sigma^2}\right) \sum_1^n x_j^2,$$

and $i = \sqrt{(-1)}$. Throughout this paper we shall understand that the limits of integration are $-\infty$ and ∞ unless otherwise specified. If we write

$$b_{11} = \sum a_{1j}^2, \quad b_{22} = \sum a_{2j}^2, \cdots, \quad b_{mm} = \sum a_{mj}^2,$$
$$b_{12} = b_{21} = \sum a_{1j}a_{2j}, \cdots, \quad b_{m-1,m} = b_{m,m-1} = \sum a_{m-1,j}a_{mj},$$

and

$$Q = \sum_{j,k} b_{jk}t_j t_k,$$

then

$$\phi(t_1, \cdots, t_{m+1}) = \frac{e^{-\sigma^2 Q/2(1-2i\sigma^2 t_{m+1})}}{[1 - 2i\sigma^2 t_{m+1}]^{n/2}}.$$

From this latter result, it is fairly obvious that the problem has no solution unless Q is a positive definite quadratic form of rank m. Upon writing $t_{m+1} = 0$, we find the characteristic function of the joint distribution of the m linear forms to be

$$\phi(t_1, \cdots, t_m, 0) = e^{-\sigma^2 Q/2}.$$

Moreover, if $\psi = \psi(u_1, \cdots, u_m)$ is the simultaneous distribution function of these linear forms, then

$$\psi = \left(\frac{1}{2\pi}\right)^m \int \cdots \int e^{-L-\sigma^2 Q/2} dt_m \cdots dt_1,$$

where

$$L = it_1u_1 + \cdots + it_mu_m.$$

Since Q is positive definite of rank m, the Cayley-Hamilton equation of the matrix $B = \|b_{jk}\|$ of Q has m real positive roots, say $\lambda_1, \lambda_2, \cdots, \lambda_m$. Moreover, there exists a real orthogonal matrix $C = \|c_{jk}\|$ such that

$$C'BC = \begin{Vmatrix} \lambda_1 & 0 & 0 \cdots 0 \\ 0 & \lambda_2 & 0 \cdots 0 \\ \cdot & & \\ \cdot & & \\ 0 & 0 \cdots 0 & \lambda_m \end{Vmatrix}.$$

If then, in the latter integral, we introduce new variables $z_1, \cdots, z_m$ by subjecting the t's to a linear homogeneous transformation with matrix C, we get

$$\psi = \left(\frac{1}{2\pi}\right)^m \int \cdots \int e^{-iS_1z_1 - \cdots - iS_mz_m - (\sigma^2/2)\Sigma\lambda_j z_j^2} dz_m \cdots dz_1$$

$$= \frac{1}{(\lambda_1 \cdots \lambda_m)^{1/2}(2\pi\sigma^2)^{m/2}} e^{-S_1^2/2\lambda_1\sigma^2 - \cdots - S_m^2/2\lambda_m\sigma^2},$$

where $S_p = \sum c_{pj}u_j$, $(p = 1, 2, \cdots, m)$.

3. *The Mathematical Expectation of* v. Let $F = F(u_1, \cdots, u_m, v)$ be the simultaneous distribution function of v and the m linear forms. Also, let $\bar{v}$ be the expected value of v for $u_1, \cdots, u_m$ assigned. Thus

$$\bar{v} = \int \frac{vF}{\psi} dv,$$

in which the limits of integration on v are here and elsewhere taken to cover all admissible values of that variable when $u_1, \cdots, u_m$ are regarded as assigned. Now

$$\phi(t_1, \cdots, t_m, t_{m+1}) = \int \cdots \int e^{L + ivt_{m+1}} F \, dv \, du_m \cdots du_1,$$

and

$$\left.\frac{\partial \phi}{\partial t_{m+1}}\right|_{t_{m+1}=0} = i \int \cdots \int ve^L F \, dv \, du_m \cdots du_1$$

$$= i \int \cdots \int ve^L \frac{F}{\psi} \psi \, dv \, du_m \cdots du_1$$

$$= i \int \cdots \int \bar{v}e^L \psi \, du_m \cdots du_1.$$

Thus

$$i\bar{v}\psi = \left(\frac{1}{2\pi}\right)^m \int \cdots \int e^{-L} \frac{\partial \phi}{\partial t_{m+1}}\bigg|_{t_{m+1}=0} dt_m \cdots dt_1.$$

But

$$\frac{\partial \phi}{\partial t_{m+1}}\bigg|_{t_{m+1}=0} = i(n\sigma^2 - Q\sigma^4)e^{-\sigma^2 Q/2}.$$

Accordingly,

$$\bar{v}\psi = \left(\frac{1}{2\pi}\right)^m \int \cdots \int (n\sigma^2 - Q\sigma^4)e^{-L-\sigma^2 Q/2} dt_m \cdots dt_1,$$

$$= n\sigma^2\psi - \sigma^2\psi\left[\left(1 - \frac{S_1^2}{\lambda_1\sigma^2}\right) + \left(1 - \frac{S_2^2}{\lambda_2\sigma^2}\right) + \cdots + \left(1 - \frac{S_m^2}{\lambda_m\sigma^2}\right)\right],$$

and

$$\bar{v} = \sigma^2\left[n - m + \frac{1}{\sigma^2}\left(\frac{S_1^2}{\lambda_1} + \cdots + \frac{S_m^2}{\lambda_m}\right)\right].$$

We now see that if each linear form is set equal to zero, the expected value of v is $\bar{v} = (n-m)\sigma^2$. Thus, when $u_1 = u_2 = \cdots = u_m = 0$, we may say that we lose one degree of freedom for each linear restriction in estimating σ^2 from v.

4. *Independent Linear Restrictions.* Of particular interest is the case in which the variables u_j are not correlated. A necessary and sufficient condition for the independence of the variables u_j is that

$$\phi(t_1, 0, \cdots, 0, 0)\cdot\phi(0, t_2, 0, \cdots, 0)\cdots\phi(0, \cdots, 0, t_m, 0) = \phi(t_1, \cdots, t_m, 0);$$

that is, when $b_{jk} \neq 0$, $j = k$, and $b_{jk} = 0$, $j \neq k$. Under these conditions, ψ becomes

$$\psi = \left(\frac{1}{2\pi\sigma^2}\right)^{m/2} \frac{1}{(b_{11}\cdots b_{mm})^{1/2}} e^{-u_1^2/2\sigma^2 b_{11} - \cdots - u_m^2/2\sigma^2 b_{mm}},$$

and

$$\bar{v} = \sigma^2\left[n - m + \frac{1}{\sigma^2}\left(\frac{u_1^2}{b_{11}} + \cdots + \frac{u_m^2}{b_{mm}}\right)\right].$$

Again we observe that the expected value of v is $(n-m)\sigma^2$ when each of the m linear forms is equated to zero. However, if s of the m linear forms are equated to their respective standard derivations while the remaining $m-s$ are equated to zero, then $\bar{v} = (n-m+s)\sigma^2$. Finally we see that the expected value of v, for a fixed set of u's, is not in general an integral multiple of σ^2.

THE UNIVERSITY OF IOWA

ON THE PRESERVATION OF ANGLES AT A BOUNDARY POINT IN CONFORMAL MAPPING†

BY S. E. WARSCHAWSKI

The object of this note is to prove the following theorem.

THEOREM. *Let R be a simply connected "schlicht" region in the w-plane whose boundary contains the point $w=0$. Let $w=0$ be "accessible" along the Jordan curve L. Suppose that there is a circle $|w|<\rho$ such that the part of the boundary of R which is inside this circle lies within the angles*

$$(1) \qquad |\arg w - h_+| \leqq k_+, \quad |\arg w - h_-| \leqq k_-, \quad (h_- \leqq h_+).$$

Suppose, furthermore, that L connects $w=0$ with a boundary point outside $|w|=\rho$ such that L divides R into two sub-regions. Let all boundary points of one sub-region which are in $|w|<\rho$, and not on L, be in one of the angles (1), and those of the other sub-region which are in $|w|<\rho$, and not on L, be in the other.

Let $w=w(z)$ map $|z-1|<1$ conformally on R in such a manner that its inverse function approaches 0 as $w\to 0$ along L. Let

$$(2) \qquad \begin{aligned} H(\alpha) &= \frac{1}{\pi}\left[\left(\frac{\pi}{2}+\alpha\right)h_+ + \left(\frac{\pi}{2}-\alpha\right)h_-\right], \\ K(\alpha) &= \frac{1}{\pi}\left[\left(\frac{\pi}{2}+\alpha\right)k_+ + \left(\frac{\pi}{2}-\alpha\right)k_-\right]. \end{aligned}$$

† Presented to Society, October 26, 1935.

Then

$$(3) \quad -\epsilon - K(\arg z) \leqq \arg w(z) - H(\arg z) \leqq K(\arg z) + \epsilon,$$

where $\epsilon \to 0$ *as* $z \to 0$ *in any fixed angle* A_Θ: $|\arg z| \leqq \Theta < \pi/2$.

Before proving this theorem we make the following remarks.

1. If the hypothesis of this theorem is satisfied for $k_+ = k_- = k$, where k is any arbitrary positive number, provided ρ is small enough, then we say that the boundary of R has a *corner of measure* $\beta = h_+ - h_-$ at $w = 0$. In this case (3) means that

$$\lim \arg \frac{w(z)}{z^{\beta/\pi}}$$

exists as $z \to 0$ in any fixed angle A_Θ.

If $\beta = \pi$, we shall say that the boundary of R has a *tangent* at $w = 0$.

2. Our theorem generalizes the well known result of C. Carathéodory† and E. Lindelöf,‡ namely, that the conformal representation of the interior of a *Jordan curve* on a circle is "quasiconform" at a boundary point with a corner of measure $\beta > 0$, that is, the angles at this boundary point are transformed proportionally. This result has been extended recently by J. Wolff§ for $\beta = \pi$ for the case where the boundary is not necessarily a Jordan curve. In 1918 W. Gross‖ obtained a theorem which infers the result of our theorem from a less general hypothesis.¶ A theorem analogous to ours for the case where the

† C. Carathéodory, Schwarz-Festschrift, pp. 40–41. See also *Conformal Representation*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 28, pp. 91–93.

‡ E. Lindelöf, Compte Rendu du quatrième congrès des mathématiciens Scandinaves, Stockholm, 1916, p. 87.

§ J. Wolff, Comptes Rendus, vol. 200 (1935), pp. 42–43.

‖ W. Gross, Mathematische Zeitschrift, vol. 2 (1918), p. 278.

¶ His hypothesis can be stated as follows (see loc. cit., pp. 276–277): Let R, the point $O(w=0)$ and L have the same meaning as in our theorem. Draw a circle c_n with center O and radius $1/n$, where $n \geqq n_0$ is so large that c_n intersects the curve L. Follow L from O up to the first point of intersection P_n with c_n. Let γ_n denote the longest open arc of c_n within R containing P_n. Every "cut" γ_n, $(n = n_0, n_0+1, \cdots)$, divides R into two regions. Call B_n that one which contains the part of L from O to P_n. Let Γ_n be the part of the boundary of B_n which B_n has in common with R. According to Carathéodory's "Primenden" theory the accessible boundary points of R can be arranged in a "cyclic order"

boundary of R in the neighborhood of O is a *Jordan curve* has been given recently by A. Ostrowski.† All results mentioned above are contained as special cases in the theorem given in this note.

For the proof of our theorem we need the following lemma.

LEMMA. *Let $u(\zeta)$ be harmonic and bounded in the circle $|\zeta-1|<1$. Suppose that there exists an arc α_+ of $|\zeta-1|=1$, starting from $\zeta=0$, in the upper half-plane and an arc α_- of $|\zeta-1|=1$, starting from $\zeta=0$, in the lower half-plane such that the radial boundary values $u^*(\zeta)$ of $u(\zeta)$ on $|\zeta-1|=1$, which exist for almost all ζ on $|\zeta-1|=1$, satisfy the conditions*

$$(4) \qquad |u^*(\zeta)-h_+| \leqq k_+ \text{ on } \alpha_+, \qquad |u^*(\zeta)-h_-| \leqq k_- \text{ on } \alpha_-,$$

where h_+, h_-, $k_+ \geqq 0$ and $k_- \geqq 0$ are given numbers. Then, if $\arg\zeta$ is the branch which is 0 for ζ real and >0,

$$(5) \qquad -\epsilon - K(\arg\zeta) \leqq u(\zeta) - H(\arg\zeta) \leqq \epsilon + K(\arg\zeta),$$

where $\epsilon \to 0$ as $\zeta \to 0$ in any way at all, and $K(\alpha)$ and $H(\alpha)$ are the functions given in (2).

PROOF. The function

$$P(\zeta) \equiv u(\zeta) - H(\arg\zeta) - K(\arg\zeta)$$

is harmonic and bounded in $|\zeta-1|<1$. Because of (4) its boundary values $P^*(\zeta)$ on $|\zeta-1|=1$ which exist for almost all values of ζ on $|\zeta-1|=1$ satisfy the relation

$$\overline{\lim_{\zeta\to 0}}\, P^*(\zeta) \leqq 0.$$

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by means of the conformal representation of R. Gross requires now that all directions from O to another accessible boundary point $P \neq O$ of Γ_n for all P "before" O be within one of the angles (1) and for all P "behind" O be in the other, provided n is sufficiently large. It is easily seen that, if this hypothesis is satisfied, then also that of our theorem holds. The converse is not necessarily true, as simple examples like the following show: R is the circle $|w-1|<1$ without the following segments: (i) $\{\arg(w-1)=\pi/2,\ 3/4 \leq |w-1| \leq 1\}$ and (ii) $\{\pi-\pi/(2n) \leq \arg(w-1) \leq \pi/2,\ |w-1|=1-1/n^2\}$, $(n=2, 3, \cdots)$; the boundary point in question is $w=0$ and L is the radius at $w=0$. Our hypothesis is evidently satisfied with $h_+=-h_-=\pi/2$ and any $k_+=k_-=k>0$, if ρ is small enough. But the hypothesis of Gross is not fulfilled.

† Acta Mathematica, vol. 64 (1935), pp. 172–174.

Furthermore, since $P(\zeta)$ is bounded, it can be represented by the Poisson integral with the boundary function $P^*(\zeta)$. Therefore, according to a theorem on Poisson's integral due to A. Ostrowski,

$$\overline{\lim_{\zeta \to 0}} P(\zeta) \leqq 0$$

as $\zeta \to 0$ in $|\zeta - 1| < 1$ in any way at all.†

This proves the inequality on the right side of (5). That on the left is obtained by applying the same argument to $-[u(\zeta) - H(\arg \zeta) + K(\arg \zeta)]$.

PROOF OF THE THEOREM. 1. Let S_ρ denote the largest connected subregion of R and of $|w| < \rho$ containing the part of L which is obtained when L is described from $O(w=0)$ up to the first point of intersection P with $|w| = \rho$. Call a the longest arc of $|w| = \rho$ within R containing P; a is part of the boundary of S_ρ. The region S_ρ is simply connected. Therefore we can map the circle $|\zeta - 1| < 1$ on S_ρ conformally by means of a function $w = \omega(\zeta)$ such that its inverse function approaches O as $w \to 0$ along L. According to a well known theorem on correspondence at the boundary in conformal mapping, there is an arc a'' on $|\zeta - 1| = 1$ which contains $\zeta = 0$ neither as an interior point nor as an end point, and which is mapped by $\omega(\zeta)$ on the arc a. The function $\omega(\zeta)$ thus has continuous boundary values on the open arc a''. Being bounded, $\omega(\zeta)$ assumes radial boundary values different from 0 at all other points of $|\zeta - 1| = 1$, except perhaps for a set of Lebesgue measure 0. Since $\omega(\zeta)$ is univalent in $|\zeta - 1| < 1$, these boundary values are necessarily different from the interior points of a. Therefore they lie within the angles (1).

2. Denote now by α_+ and α_- the two arcs of $|\zeta - 1| = 1$, respectively, extending from $\zeta = 0$ up to the end points of a'', the first starting in the upper, the second in the lower half-plane. Then we prove: *For almost all points* $\zeta = 1 + e^{i\vartheta}$ *on* α_+, $\lim_{\rho \to 1} \omega(1 + \rho e^{i\vartheta})$ *lies in the first of the angles* (1), *and for almost all* $\zeta = 1 + e^{i\vartheta}$ *on* α_-, *this limit lies in the second of the angles* (1).

In order to establish this result, we need only show that if

† See A. Ostrowski, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 36 (1927), p. 350.

ζ' and ζ'' are two points on the same arc, say α_+, at which the radial limits w' and w'' of $\omega(\zeta)$ exist and are different from 0, then w' and w'' are in the same angle. Denote by L'' the image of the part OP of L within S_ρ by means of the inverse function of $\omega(\zeta)$. The arc L'' is a Jordan arc, ending at $\zeta=0$. Therefore we can find a point ζ_1' on the radius at ζ' and a point ζ_1'' on the radius at ζ'' and a Jordan arc γ' within $|\zeta-1|<1$ joining ζ_1' and ζ_1'', such that the curve γ, formed by γ' and the two straight segments $\zeta_1'\zeta'$ and $\zeta_1''\zeta''$, does not intersect L''. The image of γ by means of $w=\omega(\zeta)$ is an arc within S_ρ joining w' and w'', which does not intersect L. Therefore w' and w'' must be in the same angle.

3. Any branch of $\arg\omega(\zeta)$ is harmonic and bounded in $|\zeta-1|<1$. According to what we have said above, $\lim_{\rho\to1}$ $\arg\omega(1+\rho e^{i\theta})$ exists for almost all points on $|\zeta-1|=1$ and, if the branch is suitably chosen, we have

$$\left|\lim_{\rho\to1}\arg\omega(1+\rho e^{i\theta})-h_+\right|\leqq k_+ \text{ on } \alpha_+,$$

and

$$\left|\lim_{\rho\to1}\arg\omega(1+\rho e^{i\theta})-h_-\right|\leqq k_- \text{ on } \alpha_-.$$

Therefore we can infer from the preceding lemma:

$$(6)\quad -\eta-K(\arg\zeta)\leqq\arg\omega(\zeta)-H(\arg\zeta)\leqq K(\arg\zeta)+\eta,$$

where $\eta\to0$ *as* $\zeta\to0$ *in* $|\zeta-1|<1$ *in any way at all.*

4. In order to obtain the connection between $\omega(\zeta)$ and $w(z)$ we first note that the inverse function of $w(z)$ maps S_ρ on a subregion S_ρ' of the circle $|z-1|<1$ and the part OP of L on an arc L' ending at $z=0$. If $z(\zeta)$ is a suitable function which maps $|\zeta-1|<1$ on S_ρ' in such a manner that its inverse function approaches 0 as $z\to0$ along L, then $\omega(\zeta)=w(z(\zeta))$. We shall establish a property of $z(\zeta)$ at $\zeta=0$.

The boundary of S_ρ' consists: (i) of a Jordan arc a' in $|z-1|<1$ which is the image of the arc a of the boundary of S_ρ and joins two points of $|z-1|=1$, different from each other and from $z=0$; (ii) of another point set Σ_ρ' in $|z-1|\leqq1$, containing $z=0$. *We prove that* S_ρ' *has a tangent at* $z=0$ *in the sense of the*

definition given in the first remark above. Let ϵ, $(0<\epsilon<\pi/2)$, be an arbitrarily given number. According to a well known theorem of Lindelöf,* $w(z)\to 0$ uniformly in the angle $|\arg z|\leqq\pi/2-\epsilon$. Therefore we can find a circle C with center $z=0$ such that the image σ of the sector σ', formed by this angle and an arc of C, by means of $w(z)$, lies inside $|w|<\rho$. Moreover, σ is inside S_ρ, as can be seen in the following way. If at least one point of the arc L' is in σ', then σ is evidently part of S_ρ. If, however, σ' does not contain any points of L', then we consider the closed Jordan curve, Γ', formed by a sub-arc OQ of L', the radius l at $z=0$, and an arc joining Q and $z=1$ inside $|z-1|<1$. Since $\lim_{z_j\to 0} w(z)=0$ on L' and on l, $w(z)\to 0$ uniformly as $z\to 0$ in the interior of Γ'.† Hence, if we join a point z_1 of L' to a point z_2 of l by an arc γ' within Γ', in a sufficiently small neighborhood of $z=0$, the image γ will join $w(z_1)$ to $w(z_2)$ and will lie in $|w|<\rho$. Since $w(z_2)$ is a point in σ, this proves that σ is in the same subregion of R as the part OP of L, that is, σ is in S_ρ.

The fact that σ is inside S_ρ shows that all boundary points of S_ρ' are outside the sector σ'. But since they certainly are within $|z-1|\leqq 1$, this proves the result about the tangent at $z=0$.

5. Now we observe that, if ρ is replaced by a smaller number r, the region S_r is a subregion of S_ρ and therefore S_r' is a subregion of S_ρ'. The boundary of S_r', as well as that of S_ρ', consists of a Jordan arc, joining two points of $|z-1|=1$, different from each other and from $z=0$, and of another point set Σ_r' containing $z=0$. The set Σ_r' is part of Σ_ρ'. Furthermore, if $\rho\to 0$, S_ρ' is contracted to the point $z=0$. *Therefore, given any* ϵ, $(0<\epsilon<\pi/2)$, *we can find a* $\rho=\rho(\epsilon)$, *such that for this* ρ *the part* Σ_ρ' *of the boundary of* S_ρ' *lies inside the angles*

$$-\frac{\pi}{2}\leqq\arg z\leqq-\frac{\pi}{2}+\epsilon, \quad \textit{and} \quad \frac{\pi}{2}-\epsilon\leqq\arg z\leqq\frac{\pi}{2}.$$

We assume that a fixed number ϵ is given, and that ρ and accordingly S_ρ is chosen in such a manner that this condition is satisfied. Then S_ρ' evidently satisfies conditions at $z=0$ analo-

* See Acta Societatis Scientiarum Fennicae, vol. 46 (1915), no. 3, p. 10.

† See, for example, L. Bieberbach, *Lehrbuch der Funktionentheorie*, vol. 2, p. 21.

gous to those of S_ρ at $w=0$ with $h_+=-h_-=\pi/2$, $k_+=k_-=\epsilon$. Therefore, we can apply the relation (6) and obtain the result that, for a suitably chosen branch of arg $z(\zeta)$ and arg ζ,

$$\overline{\lim_{\zeta\to 0}} \,|\arg z(\zeta) - \arg \zeta| \leqq \epsilon \tag{7}$$

as $\zeta\to 0$ in $|\zeta-1|<1$ in any way at all.

Let $\zeta=\zeta(z)$ denote the inverse function of $z(\zeta)$. Then $\zeta(z)\to 0$ as $z\to 0$ in any fixed angle A_Θ: $|\arg z|\leqq\Theta<\pi/2$ within S_ρ'. From this and (7) it follows easily that also

$$\overline{\lim_{z\to 0}} \,|\arg \zeta(z) - \arg z| \leqq \epsilon \tag{8}$$

as $z\to 0$ in any fixed angle A_Θ.

6. By observing that $w(z(\zeta))=\omega(\zeta)$ we obtain for z in A_Θ

$$\begin{aligned}\overline{\lim_{z\to 0}} &[\arg w(z) - H(\arg z) - K(\arg z)] \\ &\leqq \overline{\lim_{\zeta\to 0}} [\arg \omega(\zeta) - H(\arg \zeta) - K(\arg \zeta)] \\ &\quad + \overline{\lim_{z\to 0}} \,|H(\arg (\zeta(z))) - H(\arg z)| \\ &\quad + \overline{\lim_{z\to 0}} \,|K(\arg \zeta(z)) - K(\arg z)|.\end{aligned}$$

From (6) and (8) it follows that

$$\begin{aligned}\overline{\lim_{z\to 0}} &[\arg w(z) - H(\arg z) - K(\arg z)] \\ &\leqq \frac{\epsilon}{\pi}[(h_+ - h_-) + |k_+ - k_-|].\end{aligned}$$

In the same way we obtain

$$\begin{aligned}\underline{\lim_{z\to 0}} &[\arg w(z) - H(\arg z) + K(\arg z)] \\ &\geqq -\frac{\epsilon}{\pi}[(h_+ - h_-) + |k_+ - k_-|].\end{aligned}$$

Since this holds for any $\epsilon>0$ and the left side is independent of ϵ, we can let ϵ tend to 0 and obtain the result (3).

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A NOTE ON A PRECEDING PAPER*

BY FRANCIS REGAN

1. *Introduction.* In a paper† by the author, the following lemma was proved.

LEMMA. *If X_s is an element of $A(p)$, the number $\{\sum_{q=1}^{n}(q/n)X_s\vee\}$ is a member of the set $A[1-(1-p)^n]$.*

Then again,‡ the author applied a theorem due to Copeland§. It is our purpose here to extend these two theorems to apply in the field of geometrical probability. The proof of the theorem corresponding to Copeland's follows a different procedure from that given by him. As a matter of fact, the theorem of Copeland may be proved by the method given here.

2. *Extension of the Lemma.* The extension is as follows.

THEOREM 1. *If the numbers $(q/n)x(E_q)$, $(q=1, 2, \cdots, n)$, are such that $x(E_q)=\cdot\phi_{E_q}(P_1), \phi_{E_q}(P_2), \cdots$, where E_q is the interval $0<y\leqq p_q$ and $P_1, P_2, \cdots$ is a set of points admissibly ordered with respect to the function $m(E)$ (the Lebesgue measure of E) defined in Δ; $0<y\leqq 1$, then (1) the number $\sum_{q=1}^{M}(q/n)x(E_q)\vee$ has the probability $[1-\prod_{q=1}^{M}(1-p_q)]$ and (2) the number $\sum_{q=1}^{M}(q/n)x(E_q)\vee$ is a member of the set $A[1-\prod_{q=1}^{M}(1-p_q)]$, where $M\leqq n$.*

PROOF OF (1). We know that‖

$$\text{(a)}\qquad \sim\sum_{q=1}^{M}\left(\frac{q}{n}\right)x(E_q)\vee=\prod_{q=1}^{M}\sim\left(\frac{q}{n}\right)x(E_q)\cdot.$$

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* Presented to the Society, February 29, 1936.

† See the author's memoir *The application of the theory of admissible numbers to time series with constant probability*, Transactions of this Society, vol. 36 (1934), p. 517.

‡ Same reference as above, p. 524.

§ See Copeland, *Admissible numbers in the theory of probability*, American Journal of Mathematics, vol. 50 (1928), p. 550, Theorem 16.

‖ The symbol $\sum(q/n)x(E_q)\vee$ represents the number $\{(1/n)x(E_1)\vee\cdots\vee(M/n)x(E_M)\vee\}$, while $\Pi\sim(q/n)x(E_q)\cdot$ represents $\{\sim(1/n)x(E_1)\cdot\cdots\cdot(M/n)x(E_M)\cdot\}$. Throughout the paper such symbols will have similar mean-

The numbers $\sim(q/n)x(E_q)$, $(q=1, 2, \cdots, M)$, are independent, since $(q/n)x(E_q)$, $(q=1, 2, \cdots, M)$, are independent. From (a), we obtain

$$\sum_{q=1}^{M}\left(\frac{q}{n}\right)x(E_q)\vee = \sim\prod_{q=1}^{M}\sim\left(\frac{q}{n}\right)x(E_q)\cdot,$$

and hence we have

$$p\left[\sum_{q=1}^{M}\left(\frac{q}{n}\right)x(E_q)\vee\right] = 1 - p\left[\prod_{q=1}^{M}\sim\left(\frac{q}{n}\right)x(E_q)\cdot\right],$$

or

$$p\left[\sum_{q=1}^{M}\left(\frac{q}{n}\right)x(E_q)\vee\right] = 1 - \prod_{q=1}^{M}(1-p_q),$$

where $M\leqq n$.

PROOF OF (2). Since

$$\sum_{q=1}^{M}\left(\frac{q}{n}\right)x(E_q)\vee = \sim\prod_{q=1}^{M}\sim\left(\frac{q}{n}\right)x(E_q)\cdot,$$

then

$$\left(\frac{r_i}{m}\right)\left[\sum_{q=1}^{M}\left(\frac{q}{n}\right)x(E_q)\vee\right] = \sim\prod_{q=1}^{M}\sim\left(\frac{[q+(r_i-1)n]}{mn}\right)x(E_q)\cdot.$$

Then

$$\prod_{i=1}^{k}\left(\frac{r_i}{m}\right)\left[\sum_{q=1}^{M}\left(\frac{q}{n}\right)x(E_q)\vee\right] = \prod_{i=1}^{k}\sim\prod_{q=1}^{M}\sim\left(\frac{[q+(r_i-1)n]}{mn}\right)x(E_q)\cdot.$$

The numbers r_i are chosen such that for every set $r_1, r_2, \cdots, r_k$, we have $0<r_i\leqq m$ and $r_i\neq r_j$ if $i\neq j$. The numbers $\sim([q+(r_i-1)n]/mn)x(E_q)$ are independent. Hence the numbers $\prod_{q=1}^{M}\sim([q+(r_i-1)n]/mn)x(E_q)\cdot$ are independent, from

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ings. For the truth of this equality, see Copeland, *The theory of probability from the point of view of admissible numbers*, Annals of Mathematical Statistics, vol. 3 (1932), p. 149.

which it follows that the numbers $\sim\prod_{q=1}^{M}\sim([q+(r_i-1)n]/mn)\,x(E_q)\cdot$ are independent. We may now conclude that

$$p\left\{\prod_{i=1}^{k}\left(\frac{r_i}{m}\right)\left[\sum_{q=1}^{M}\left(\frac{q}{n}\right)x(E_q)\vee\right]\cdot\right\}=\left\{1-\prod_{q=1}^{M}(1-p_q)\right\}^{k}.$$

Therefore, the number $\sum_{q=1}^{M}(q/n)x(E_q)$ is an element of $A[(1-\prod_{q=1}^{M}(1-p_q))]$, where $M\leqq n$.

3. *Analog of Copeland's Theorem.* In order to prove the second theorem, we shall need the following lemma.

LEMMA. *If the numbers* $x_1^1, x_2^1, \cdots, x_{N_1}^1, x_1^2, x_2^2, \cdots, x_{N_2}^2, \cdots, x_1^k, x_2^k, \cdots, x_{N_k}^k$ *are such that* $x_j^i\cdot x_{j'}^i=0$, *where* $j\vee j'$ *and* $x_{\gamma_1}^1, x_{\gamma_2}^2, \cdots, x_{\gamma_k}^k$ *are independent, it follows that the numbers* $(x_1^1\vee x_2^1\vee\cdots\vee x_{N_1}^1)$, $(x_1^2\vee x_2^2\vee\cdots\vee x_{N_2}^2), \cdots, (x_1^k\vee x_2^k\vee\cdots\vee x_{N_k}^k)$ *are independent.*

By hypothesis, $x_1^1, x_{\gamma_2}^2, \cdots, x_{\gamma_k}^k$ and $x_2^1, x_{\gamma_2}^2, \cdots, x_{\gamma_k}^k$ are independent, and since $x_1^1\cdot x_2^1=0$, the numbers $x_1^1\vee x_2^1$, $x_{\gamma_2}^2, \cdots, x_{\gamma_k}^k$ are independent.* Then the two sets of numbers $x_1^1\vee x_2^1, x_{\gamma_2}^2, \cdots, x_{\gamma_k}^k$ and $x_3^1, x_{\gamma_2}^2, \cdots, x_{\gamma_k}^k$ are independent, and since $(x_1^1\vee x_2^1)\cdot x_3^1=(x_1^1\cdot x_3^1)\vee(x_2^1\cdot x_3^1)=0$, the numbers $x_1^1\vee x_2^1\vee x_3^1, x_{\gamma_2}^2, \cdots, x_{\gamma_k}^k$ are independent. In general, the numbers $(x_1\vee x_2^1\vee\cdots\vee x_{N_1}^1), x_{\gamma_2}^2, \cdots, x_{\gamma_k}^k$ are independent.

Applying the above to each of the $(k-1)$ remaining groups of numbers, we conclude that the numbers $(x_1\vee x_2\vee\cdots\vee x_{N_1}^1)$, $(x_1^2\vee x_2^2\vee\cdots\vee x_{N_2}^2), \cdots, (x_1^k\vee x_2^k\vee\cdots\vee x_{N_k}^k)$ are independent numbers. Hence we have proved the lemma.

We now come to the analog of the theorem of Copeland.

THEOREM 2. *If the numbers* $(q/n)x(E_q)$, $(q=1, 2, \cdots, n)$, *are such that* $x(E_q)=\cdot\phi_{Eq}(P_1), \phi_{Eq}(P_2)\cdots$, *where* E_q *is the interval* $0<y\leqq p_q$ *and* $P_1, P_2, \cdots$ *is admissibly ordered with respect to the function* $m(E)$ *defined in* Δ: $0<y\leqq 1$, *and if*

$$X=\sum_{j=1}^{M}Y_j\vee \text{ and } Y_j=\prod_{i=1}^{\alpha_j}\left(\frac{q_{ij}}{n}\right)x(E_{q_{ij}})\cdot\prod_{i=\alpha_j+1}^{n}\sim\left(\frac{q_{ij}}{n}\right)x(E_{q_{ij}})\cdot,$$

where $0<q_{ij}\leqq n$ *and* $q_{i'j}\neq q_{ij}$ *if* $i'\neq i$, *and where* $Y_{j'}\neq Y_j$ *if* $j'\neq j$, *then* X *belongs to the set* $A(P)$, *where*

* See Copeland, *Admissible numbers in the theory of probability*, American Journal of Mathematics, vol. 50 (1928), p. 543, Theorem 6.

$$P = \sum_{j=1}^{M} \left\{ \prod_{i=1}^{\alpha_j} p_{q_{ij}} \cdot \sum_{i=\alpha_j+1}^{n} (1 - p_{q_{ij}}) \right\}.$$

Since the numbers $(q/n)x(E_q)$ are independent, the numbers $\sim(q/n)x(E_q)$ are independent. By hypothesis, we know that $Y_j \cdot Y_{j'} = 0$ if $j \neq j'$. Hence $p(X) = P$. Now we wish to show that

$$p\left[\prod_{s=1}^{k} \left(\frac{r_s}{m}\right) X \cdot \right] = P^k$$

for every positive integer m and for every set of distinct integers $r_1, r_2, \cdots, r_k$, such that $0 < r_s \leqq m$. We know that

$$\left(\frac{r_s}{m}\right) Y_j = \prod_{i=1}^{\alpha_j} \left(\frac{[q_{ij} + (r_s - 1)n]}{mn} \right) x(E_{q_{ij}}) \cdot \prod_{i=\alpha_j+1}^{n} \sim \left[\frac{[q_{ij} + (r_s - 1)n]}{mn} \right] x(E_{q_{ij}}) \cdot .$$

The numbers constituting the above product are independent, and moreover $(r_1/m) Y_{j_1}$, $(r_2/m) Y_{j_2}, \cdots, (r_k/m) Y_{j_k}$ are independent regardless of whether $j_1, j_2, \cdots, j_k$ are equal or not. Within each group any two distinct numbers are mutually exclusive; that is, $(r_s/m) Y_j \cdot (r_s/m) Y_{j'} = 0$ if $j \neq j'$. We may now apply the above lemma. Hence the numbers

$$\sum_{j=1}^{M} (r_1/m) Y_j \vee, \quad \sum_{j=1}^{M} (r_2/m) Y_j \vee, \cdots, \quad \sum_{j=1}^{M} (r_k/m) Y_j \vee$$

are independent. We know that

$$p\left[\prod_{s=1}^{k} \left(\frac{r_s}{m}\right) X \cdot \right] = p\left[\prod_{s=1}^{k} \left(\frac{r_s}{m}\right) \left\{ \sum_{j=1}^{M} Y_j \vee \right\} \cdot \right] = p\left[\prod_{s=1}^{k} \left\{ \sum_{j=1}^{M} \left(\frac{r_s}{m}\right) Y_j \vee \right\} \cdot \right];$$

but since the numbers $\sum_{j=1}^{M} (r_s/m) Y_j \vee$ are independent, the last term is equal to P^k. Therefore the theorem is proved.

It is obvious that the above theorems can be extended to an n-dimensional continuum.

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ON THE PRINCIPLES OF HAMILTON AND CARTAN
(SUPPLEMENTARY NOTE)

BY J. W. CAMPBELL

A request from Dr. D. C. Lewis for an explanation regarding a transformation which appeared in my paper on this topic (this Bulletin, February, 1936) led me on examination to find that the results there given are true in a restricted case but not in general. They are valid when the a_{rs} and the a_s of the non-holonomic relations are functions of t only. Also, under these conditions it can be shown that the order of integration in (15) and (16) is reversible and that then the action integral

$$\int_{u_0}^{u_1} \sum_1^k p_r dq_r - H dt$$

is an extremal for an arc of a trajectory as compared with all neighboring paths which are kinematically possible.

There is, however, a coordination of the principles for the general case. For, using the notation which was adopted, consider the expression

$$\begin{aligned} \sum_1^k \Big\{ &\Big(dq_r - \frac{\partial H}{\partial p_r} dt \Big) \delta p_r \\ &+ \Big(- dp_r - \frac{\partial H}{\partial q_r} dt + \sum_1^m a_{rs} \lambda_s dt \Big) \delta q_r \\ &+ \Big(dH - \frac{\partial H}{\partial t} dt + \sum_1^m a_s \lambda_s dt \Big) \delta t \Big\}. \end{aligned} \tag{17}$$

The integral of (17) over a Cartan locus can, after an integration by parts, be written as the negative of

$$d \int_{\alpha_0}^{\alpha_1} \sum_1^k p_r \delta q_r - H \delta t - \int_{\alpha_0}^{\alpha_1} \sum_1^m \lambda_s \Big(\sum_1^k a_{rs} \delta q_r + a_s \delta t \Big) dt, \tag{18}$$

and its integral over an arc of a trajectory is

$$\delta \int_{u_0}^{u_1} \sum_1^k p_r dq_r - H dt + \int_{u_0}^{u_1} \sum_1^m \lambda_s \Big(\sum_1^k a_{rs} \delta q_r + a_s \delta t \Big) dt. \tag{19}$$

But if we consider a cylinder of trajectories through a Cartan locus and consider the variations from the trajectories as being confined to the surface of such a cylinder, then (18) and (19) may be written

$$d\left\{\int_{\alpha_0}^{\alpha_1}\sum_1^k p_r\delta q_r - H\delta t - \int_{\alpha_0}^{\alpha_1}\int_{t_0}^{t}\sum_1^m \lambda_s\left(\sum_1^k a_{rs}\delta q_r + a_s\delta t\right)dt\right\}, \tag{20}$$

and

$$\delta\left\{\int_{u_0}^{u_1}\sum_1^k p_r dq_r - Hdt + \int_{u_0}^{u_1}\int_{\alpha_0}^{\alpha}\sum_1^m \lambda_s\left(\sum_1^k a_{rs}\delta q_r + a_s\delta t\right)dt\right\}, \tag{21}$$

where $\alpha=\alpha(u, v)$, v a parameter, is arbitrary except that

$$\alpha(u, 0) \underset{u}{\equiv} \alpha_0, \qquad \frac{\partial\alpha(u_0, v)}{\partial v} \underset{v}{\equiv} 0, \qquad \frac{\partial\alpha(u_1, v)}{\partial v} \underset{v}{\equiv} 0.$$

The vanishing of (17) identically in δq_r, δp_r, and δt therefore implies, and is implied by, the vanishing of either (20) or (21), for since the cylinder and the law of the Cartan locus given by ρ are arbitrary, the principle of the identical vanishing of coefficients is valid.

Hence, since the λ's are functions of t which are not identically zero, a non-holonomic system which is defined by (5) and (6) of my paper is characterized either by the Cartan integral invariant

$$\int_{\alpha_0}^{\alpha_1}\left\{\sum_1^k p_r\delta q_r - H\delta t - \int_{t_0}^{t}\sum_1^m \lambda_s\left(\sum_1^k a_{rs}\delta q_r + a_s\delta t\right)dt\right\}, \tag{22}$$

or by the Hamilton extremal integral

$$\int_{u_0}^{u_1}\left\{\sum_1^k p_r dq_r - Hdt + \int_{\alpha_0}^{\alpha}\sum_1^m \lambda_s\left(\sum_1^k a_{rs}\delta q_r + a_s\delta t\right)dt\right\}. \tag{23}$$

This is Taylor's extension and the corresponding extension of Hamilton's principle for the general case. It might be of interest to note the geometrical significance of each. In the Cartan

principle an integral over a Cartan locus is varied by slipping the locus over the cylinder, points going along trajectories, while in the Hamilton principle the same expression is integrated along an arc of a trajectory and variation takes place by keeping the end points fixed but slipping the intervening path over the surface of a cylinder, points going over arcs of Cartan loci.

Finally, two points may be noted.

(1) From the form of (23) Whittaker's remarks which were cited follow as a corollary.

(2) Since (23) does not reduce to (16) as a special case, there are for the restricted conditions to which the original paper applies two different characterizing Hamilton extremal integrals.

THE UNIVERSITY OF ALBERTA

NOTE ON THE CANONICAL FORM OF THE PARAMETRIC EQUATIONS OF A SPACE CURVE BELONGING TO A NON-SPECIAL LINEAR LINE COMPLEX

BY C. R. WYLIE, JR.

In a recent paper,* the author, by means of a projection from hyper-space, obtained the following equations for a general curve belonging to a linear complex,

$$\text{A:} \qquad x_1 = -t, \qquad x_2 = f - \frac{1}{2}tf', \qquad x_3 = -f', \qquad x_4 = 1.$$

It is the purpose of this note to call attention to a more symmetric form of these equations.

Let $x_i = f_i(s)$ be the equation of a general space curve, and $P_{13} = P_{42}$ be the equation of a general linear complex. If the curve belongs to the complex

$$f_1f_3' - f_3f_1' = f_4f_2' - f_2f_4', \text{ or } \frac{f_1^2(f_1f_3' - f_3f_1')}{f_4^2 \cdot f_1^2} = \frac{(f_4f_2' - f_2f_4')}{f_4^2};$$

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* C. R. Wylie, Jr., *Space curves belonging to a non-special linear line complex*, American Journal of Mathematics, vol. 57 (1935), pp. 937–942.

this last can be written in the form

$$\left(\frac{f}{f_4}\right)^2 \cdot u' = v',$$

where $u = f_3/f_1$ and $v = f_2/f_4$. From these we have at once

B.1: $f_1 = \left(\frac{v'}{u'}\right)^{1/2} \cdot f_4,\ f_2 = v \cdot f_4,\ f_3 = u\left(\frac{v'}{u'}\right)^{1/2} \cdot f_4,\ f_4 = f_4,$

B.2: $x_1 = (v')^{1/2},\quad x_2 = v(u')^{1/2},\quad x_3 = u(v')^{1/2},\quad x_4 = (u')^{1/2}.$

To show the equivalence of A and B.2, make the substitutions

C.1: $\left(\frac{v'}{u'}\right)^{1/2} = t,$ C.2: $\frac{df}{dt} = u\left(\frac{v'}{u'}\right)^{1/2} = ut.$

From C.1 and C.2 we obtain after an easy integration by parts

C.3: $v = t \cdot \frac{df}{dt} - 2f.$

Using these expressions in B.1 we find

D: $x_1 = t,\quad x_2 = tf' - 2f,\quad x_3 = f',\quad x_4 = 1,$

where now the primes indicate differentiation with respect to t instead of s. The slight difference between D and A is due to the fact that in the paper mentioned above, the equation of the complex to which the curve belongs was taken to be $P_{13} + 2P_{42} = 0$.

OHIO STATE UNIVERSITY

ON THE NON-VANISHING OF THE JACOBIAN IN CERTAIN ONE-TO-ONE MAPPINGS

BY HANS LEWY

THEOREM 1. *If $u(x, y)$ and $v(x, y)$ are harmonic, $u(0, 0) = v(0, 0) = 0$, and if there exists a neighborhood N_1 of the origin of the xy plane and a neighborhood N_2 of the origin of the uv plane such that $u(x, y)$ and $v(x, y)$ establish a mapping of N_1 onto N_2 which is one-to-one both ways, then the Jacobian $\partial(u, v)/\partial(x, y)$ does not vanish at the origin.*

PROOF. As the statement of Theorem 1 remains invariant under homogeneous linear transformations of the uv plane, we may assume, in the developments in polar coordinates for u and v, that

$$u = \sum_{i}^{\infty} [a_n r^n \cos n\theta + b_n r^n \sin n\theta], \qquad (a_i^2 + b_i^2 \neq 0),$$

$$v = \sum_{k}^{\infty} [A_n r^n \cos n\theta + B_n r^n \sin n\theta], \qquad (A_k^2 + B_k^2 \neq 0),$$

that the positive index i does not exceed k, and that for $i = k$ we have $a_k B_k - A_k b_k \neq 0$. Considering the case $i = k$ first, we may, because of the invariance mentioned, assume

$$a_k = B_k = 1, \qquad b_k = A_k = 0.$$

For small values of r, the auxiliary mapping,

$$\bar{u} = r^k \cos k\theta, \qquad \bar{v} = r^k \sin k\theta,$$

can be continuously joined with the given one for $0 \leqq t \leqq 1$ by

$$u_t = r^k \cos k\theta + t \sum_{k+1}^{\infty} (a_n r^n \cos n\theta + b_n r^n \sin n\theta),$$

$$v_t = r^k \sin k\theta + t \sum_{k+1}^{\infty} (A_n r^n \cos n\theta + B_n r^n \sin n\theta).$$

The vector (u_t, v_t) thereby never differs by more than a vector of length $r^k/2$ from the vector $(\bar{u}, \bar{v})$ whose length is r^k. Thus

the index of the origin* in both fields (u, v) and $(\tilde{u}, \tilde{v})$ is the same, and as it is ± 1 for one-to-one mappings we conclude $i = k = 1$ and our theorem follows.

In the remaining case $i < k$ we consider the Jacobian

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{r} \frac{\partial(u, v)}{\partial(r, \theta)}.$$

We find by elementary computation that the development of J starts with the terms of lowest degree in r

$$kir^{i+k-1}\left[(A_k a_i + B_k b_i) \sin (i - k)\theta + (a_i B_k - b_i A_k) \cos (i - k)\theta\right]$$

or, with a suitable angle θ_0, with the term

$$kir^{i+k-1}(A_k^2 + B_k^2)^{1/2}(a_i^2 + b_i^2)^{1/2} \cos \left[(i - k)\theta - \theta_0\right].$$

Hence, for sufficiently small values of r, J assumes both positive and negative values, while in a one-to-one map the Jacobian, where it does not vanish, is of the same sign as the index of the map which is either $+1$ everywhere or -1 everywhere. Hence $i < k$ is impossible and our proof is completed.

THEOREM 2. *Let $u(x, y)$ and $v(x, y)$ be analytic functions of x and y in a neighborhood N_1 of the origin of the xy plane which they map onto a neighborhood N_2 of the origin of the uv plane in a one-to-one correspondence. Suppose, moreover, that u and v are solutions of the following equations:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + b\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) + c\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + d\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) = 0, \tag{1}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + A\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + B\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) + C\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + D\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) = 0, \tag{2}$$

* For the topological notions used, see W. Fenchel, *Elementare Beweise und Anwendungen einiger Fixpunktsätze*, Matematisk Tidsskrift, (B), 1932, p. 66.

in which $a(u, v)$, $b(u, v)$, $\cdots$, $D(u, v)$ are analytic functions of u and v defined for (u, v) in N_2. Then the Jacobian $\partial(u, v)/\partial(x, y)$ does not vanish at the origin.

PROOF. Suppose, without loss of generality, that the power series for u and v in x and y start with non-vanishing terms of ith and kth degree, respectively, and that furthermore $i \leqq k$, $i > 0$. Then the terms of ith degree in $u(x, y)$ form a harmonic function, because this is trivial for $i = 1$, and for $i > 1$ the only terms in the development of the left hand of (1) of degree $i-2$ are furnished by those of $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$. Introduce $V(x, y) = v(x, y) - F(u)$, where F is analytic in u and $F(0) = 0$. Instead of (1) and (2) we find

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + [a(u, V + F(u)) + F'(u)b(u, V + F(u)) + F'^2(u)c(u, V + F(u))]\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + \cdots = 0, \tag{3}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + F'(u)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + [A(u, V + F(u)) + F'(u)B(u, V + F) + F'^2(u)C(u, V + F) + F''(u)]\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + \cdots = 0, \tag{4}$$

and, by substitution of (3) in (4),

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + [A(u, V + F) + F'B + F'^2C + F'' - F'a - F'^2b - F'^3c]\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + \cdots = 0, \tag{5}$$

where the omitted terms are linear in

$$\frac{\partial u}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial V}{\partial y}, \quad \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right), \quad \text{and} \quad \frac{\partial(u, V)}{\partial(x, y)}.$$

Now notice that V vanishes for $x = y = 0$ and that in its development the lowest degree K of non-vanishing terms still is $\geqq i$.

Let us distinguish two cases, $k > i$ and $k = i$. For $k > i$, we con-

struct a function $F(u)$, solution of the ordinary differential equation of second order

$$A(u, F(u)) + F'(u)B(u, F(u)) + F'^2C - F'a - F'^2b - F'^3c + F'' = 0, \tag{6}$$

vanishing with its first derivative for $u=0$. We have $k>i$, as $F(u)$, considered as function of x and y starts with terms of degree $\geqq 2i$. Equation (5) shows now a coefficient of $(\partial u/\partial x)^2+(\partial u/\partial y)^2$ whose development in u and V has *all terms divisible* by V. Thus the x, y series for the left hand of (5) obtains all terms of degree $K-2$ from the expression $\partial^2 V/\partial x^2+\partial^2 V/\partial y^2$, whence we conclude that in V the terms of lowest order K form a harmonic function. As in the proof of Theorem 1, we conclude that the Jacobian $\partial(u, V)/\partial(x, y)$, which equals $\partial(u, v)/\partial(x, y)$, would assume both positive and negative values in every neighborhood of the origin, which leads to a contradiction. Thus $i<k$ is impossible.

In the case $i=k$ we again choose $F(u)$ as solution of (6), vanishing for $u=0$, but we try to determine $F'(u)$ in such a way that k results $>i$. If this were possible, we could apply the same argument as at the end of the last paragraph, and find the same contradiction. Hence, should $i=k=1$, we would have the desired inequality

$$\frac{\partial(u, V)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(x, y)} \neq 0$$

at the origin. If, however, $i=k>1$, we may determine the linear term of $F(u)$ such that for polar coordinates in the xy plane and suitable determination of $\theta=0$ we have

$$u = \text{const. } r^k \cos k\theta + r^{k+1}(\cdots),$$
$$V = \text{const. } r^k \sin k\theta + r^{k+1}(\cdots).$$

This leads, by the same reasoning as in the proof of Theorem 1, to the conclusion that the index of the vector field (u, V) at the origin is $k\neq 1$. But together with the functions $u(x, y)$, $v(x, y)$, the functions $u(x, y)$, $V(x, y)=v(x, y)-F(u)$ also establish a one-to-one mapping of the xy plane, because the correspondence between the uv plane and the uV plane evidently is one-to-one. Hence the index ought to be ± 1, which completes our proof.

THE UNIVERSITY OF CALIFORNIA

A GENERALIZED INVERSIVE ALGORITHM

BY D. H. LEHMER

In a previous note* we considered an algorithm for constructing the nth member of any infinite class C of positive integers, an algorithm used by Brun† for the non-tentative calculation of the nth prime. In this note we give a generalization in two directions of the Brun algorithm. In the first place it is made applicable to any denumerable set of elements and secondly a parameter (m_0) is introduced which allows the algorithm to start from a more advantageous position than in the case of Brun, where one starts by saying that the nth prime is not less than n.

The algorithm may be presented in the following form:

Let $S: e_1, e_2, e_3, \cdots$, be a class of ordered elements.

Let $C: e_{a_1}, e_{a_2}, e_{a_3}, \cdots, (a_i < a_{i-1})$, be any infinite subclass of S. For x an integer‡ >0, we define $\theta(x)$ as the number of elements of C belonging to the set of the first x elements of S. For $x \leqq 0$, we define $\theta(x) = 0$. Let m_0 and n be any positive integers. We then form the sequence

$$m_1, m_2, m_3, \cdots, \tag{1}$$

defined by

$$\begin{aligned} m_1 &= n - \theta(m_0), \\ m_2 &= n - \theta(m_0 + m_1), \\ m_3 &= n - \theta(m_0 + m_1 + m_2), \\ &\cdots\cdots\cdots\cdots \\ m_r &= n - \theta(s_r), \end{aligned} \tag{2}$$

where we have written s_r for $m_0 + m_1 + \cdots + m_{r-1}$. We may then state the following theorem.

THEOREM. *The terms of* (1) *do not differ in sign, nor do they increase in absolute value. There exists moreover an integer k such*

* This Bulletin, vol. 38 (1932), pp. 693–694.

† Kongelige Norske Videnskaps-selskabet, vol. 4, pp. 66–69.

‡ If x is not an integer one could define $\theta(x) = \theta([x])$.

that $m_\nu \neq 0$ for $\nu < k$, while $m_\nu = 0$ for $\nu \geqq k$. If $m_1 > 0$, then e_{s_k} is the nth member of the class C. If $m_1 < 0$, then $e_{s_{k-1}}$ is the $(n - m_{k-1})$st member of C. If $m_1 = 0$, the nth member of C is e_h, where $n \leqq h \leqq m_0$.

PROOF. If $m_1 = 0$, then equations (2) are all identical and reduce to

$$n = \theta(m_0), \tag{3}$$

all the other m's are zero. The theorem then follows at once with $k = 1$, the last statement following from (3) which tells us that n members of C have subscripts $\leqq m_0$.

From the definition of $\theta(x)$, we have for $x > y$,

$$0 \leqq \theta(x) - \theta(y) \leqq x - y. \tag{4}$$

For $m_1 \neq 0$ we consider the difference

$$m_{r-1} - m_r = \theta(s_r) - \theta(s_{r-1}) = \theta(s_{r-1} + m_{r-1}) - \theta(s_{r-1}). \tag{5}$$

Using (4) we see that if $r \geqq 2$,

$$m_{r-1} = 0 \text{ implies } m_r = 0, \tag{6}$$

$$m_{r-1} > 0 \text{ implies } 0 \leqq m_{r-1} - m_r \leqq m_{r-1}, \tag{7}$$

$$m_{r-1} < 0 \text{ implies } 0 \geqq m_{r-1} - m_r \geqq m_{r-1}. \tag{8}$$

From these equations we see that the m's do not differ in sign. Also (7) shows that the m's do not increase if positive, while (8) shows that they do not decrease if negative. Therefore the sequence $|m_r|$, $(r = 1, 2, 3, 4, \cdots)$, is non-increasing. If possible suppose that $|m_r| > 0$, for all $r > 0$; then $s_r = m_0 + m_1 + m_2 + \cdots + m_{r-1}$ would either increase or decrease indefinitely according as $m_1 > 0$ or < 0. In the first case, since C is an infinite class, $\theta(s_r)$ would also increase without limit, so that for r sufficiently large $m_r = n - \theta(s_r)$ would be negative, contrary to fact. In the second case s_r would ultimately become negative so that $\theta(s_r) = 0$. For such an r we would have $m_r = n$. But n is positive, while m_r is not. Hence $|m_r|$ becomes and remains zero for all sufficiently large r, say $r \geqq k$.

To complete the proof of the theorem set $r = k$ in (5) and obtain

$$m_{k-1} = \theta(s_{k-1} + m_{k-1}) - \theta(s_{k-1}). \tag{9}$$

If m_1 and hence m_{k-1} is positive, (9) tells us that all the m_{k-1} members of S: $e_{s_{k-1}+1}, \cdots, e_{s_{k-1}+m_{k-1}}$ belong to C. In particular the last one which is e_{s_k} belongs to C. Moreover $m_k = n - \theta(s_k) = 0$. Hence e_{s_k} is the nth member of C. If m_1 and hence m_{k-1} is negative, (9) tells us that all the m_{k-1} members of S: $e_{s_k+1}, \cdots, e_{s_k-1}$ belong to C. In particular e_{s_k-1} is a member of C. But $m_{k-1} = n - \theta(s_{k-1})$ or $\theta(s_{k-1}) = n - m_{k-1}$; hence e_{s_k-1} is the $(n-m_{k-1})$st member of C. Thus the proof is complete.

If S and C are wholly arbitrary one may show by means of examples that the conclusion of the theorem is the best that can be obtained from the hypothesis. Additions to the hypothesis make the last two statements of the theorem more precise. For example, if we assume that no two consecutive members of C are consecutive members of S, it is easy to prove that e_{s_k-1} is the $(n-1)$st member of C in case $m_1 < 0$.

LEHIGH UNIVERSITY

A NEW CLASS OF FUNCTIONS OF TWO VARIABLES INVOLVING BESSEL FUNCTIONS OF HALF AN ODD INTEGER*

BY N. A. HALL

The evaluation of certain integrals arising in the theory of the conduction of heat between two media of different conductivities† suggested the consideration of the expansions:

$$\sin\left[\alpha(1+x^2)^{1/2} - \beta x\right] = \sum_{n=0}^{\infty} x^n S_n(\alpha, \beta), \tag{1}$$

$$\cos\left[\alpha(1+x^2)^{1/2} - \beta x\right] = \sum_{n=0}^{\infty} x^n C_n(\alpha, \beta), \tag{2}$$

$$\exp i\left[\alpha(1+x^2)^{1/2} - \beta x\right] = \sum_{n=0}^{\infty} x^n E_n^{(1)}(\alpha, \beta), \tag{3}$$

$$\exp -i\left[\alpha(1+x^2)^{1/2} - \beta x\right] = \sum_{n=0}^{\infty} x^n E_n^{(2)}(\alpha, \beta). \tag{4}$$

* Presented to the Society, November 30, 1935.

† G. Green, Philosophical Magazine, (7), vol. 18 (1934), p. 631.

The four sets of functions appearing in these series are expressible as polynomials in the Bessel functions of half an odd integer and have a number of fairly significant properties.

As is evident from the nature of the generating functions all four sets satisfy the following equations:

$$\frac{\partial^2 u_n}{\partial \alpha^2} - \frac{\partial^2 u_n}{\partial \beta^2} + u_n = 0, \tag{5}$$

$$\frac{\partial^2 u_n}{\partial \beta^2} + u_{n-2} = 0, \tag{6}$$

$$\frac{\partial^2 u_n}{\partial \alpha^2} + u_n + u_{n-2} = 0, \tag{7}$$

together with the special equations

$$\frac{\partial}{\partial \beta} C_n(\alpha, \beta) = S_{n-1}(\alpha, \beta), \tag{8}$$

$$\frac{\partial}{\partial \beta} S_n(\alpha, \beta) = - C_{n-1}(\alpha, \beta), \tag{9}$$

and similar relations between $E_n^{(1)}$ and $E_n^{(2)}$ obtainable from the above by means of

$$E_n^{(1)}(\alpha, \beta) = C_n(\alpha, \beta) + iS_n(\alpha, \beta), \tag{10}$$

$$E_n^{(2)}(\alpha, \beta) = C_n(\alpha, \beta) - iS_n(\alpha, \beta). \tag{11}$$

The equations (5)-(11) together with the known values of the functions for $\alpha = 0$ and $n = 1$ are sufficient to determine the functions uniquely. Solving these equations, we obtain the results:

$$E_n^{(1)}(\alpha, \beta) = (- i\beta)^n \left(\frac{\alpha\pi}{2}\right)^{1/2} \sum_{k=0}^{\leqq n/2} \sigma_{n,k} \left(\frac{\alpha}{2\beta^2}\right)^k H_{k-1/2}^{(1)}(\alpha), \tag{12}$$

$$E_n^{(2)}(\alpha, \beta) = (i\beta)^n \left(\frac{\alpha\pi}{2}\right)^{1/2} \sum_{k=0}^{\leqq n/2} \sigma_{n,k} \left(\frac{\alpha}{2\beta^2}\right)^k H_{k-1/2}^{(2)}(\alpha), \tag{13}$$

where

$$1/\sigma_{n,k} = \Gamma(n - 2k + 1)\Gamma(k + 1),$$

and where $H_k^{(1)}(\alpha)$ and $H_k^{(2)}(\alpha)$ are Bessel functions of the third

kind, together with expressions for S_n and C_n in terms of Bessel functions of the first and second kind. From the properties of the Bessel functions, we may also obtain expressions of the form

$$(14)\quad S_n(\alpha,\beta)=\beta^n\sum_{l=0}^{\leq n/2}\sum_{k=0}^{l-1}\frac{\sigma_{n,l}\Gamma(l+k)}{\Gamma(l-k)\Gamma(k+1)}\left(\frac{\alpha}{2\beta^2}\right)^l(2\alpha)^{-k}\sin\left(\alpha-(n+l+k)\frac{\pi}{2}\right),$$

and

$$(15)\quad \begin{aligned}C_n(\alpha,\beta)&=\sin\left(\alpha-n\frac{\pi}{2}\right)\sum_{k=0}^{\leq n/2}\sigma_{n,k}\left(\frac{\alpha}{2\beta^2}\right)^k R_{k-1,1/2}(\alpha)\\&\quad-\cos\left(\alpha-n\frac{\pi}{2}\right)\sum_{k=0}^{\leq n/2}\sigma_{n,k}\left(\frac{\alpha}{2\beta^2}\right)^k R_{k-2,3/2}(\alpha),\end{aligned}$$

where $R_{m,n}(\alpha)$ is the Lommel polynomial.*

From the well known infinite integrals involving the Bessel functions, a number of polynomials may be expressed in terms of infinite integrals involving these new functions. Among the most interesting are the expressions for the truncated hyperbolic sine and cosine series:

$$(16)\quad \sum_{\tau=0}^{n}\frac{\beta^{2\tau+1}}{(2\tau+1)!}=(-)^{n+1}\frac{\cosh^{2n+1}\tau}{\sinh\tau}\int_0^\infty e^{-\alpha\sinh\tau}S_{2n+1}(\alpha,\beta\operatorname{sech}\tau)d\alpha,$$

$$(17)\quad \sum_{\tau=0}^{n}\frac{\beta^{2\tau}}{(2\tau)!}=(-)^{n}\frac{\cosh^{2n}\tau}{\sinh\tau}\int_0^\infty e^{-\alpha\sinh\tau}C_{2n}(\alpha,\beta\operatorname{sech}\tau)d\alpha.$$

Of a more complicated nature are

$$(18)\quad \int_0^\infty E_n^{(1)}(\alpha,\beta)d\alpha=(-i)^{n+1}\frac{\beta^n}{n!}\,{}_2F_0\left(-\frac{n}{2},-\frac{n-1}{2},\frac{1}{2};\frac{4}{\beta^2}\right),$$

and

$$(19)\quad \int_0^\infty e^{-\lambda^2\alpha^2}C_{2n}(\alpha,\beta)d\alpha=\frac{\pi e^{-1/4\lambda^2}}{n!(2\lambda)^{2n+1}}\,{}_1F_1\left(-n;\frac{1}{2};\frac{\beta^2}{\lambda^2}\right).$$

In addition to the above series expansions for the functions,

* G. A. Watson, *Theory of Bessel Functions*, p. 296.

we may obtain from them or direct from the generating functions expressions in terms of contour integrals. Making use of the contour integrals for the Bessel functions* we obtain results of the type:

$$(20) \qquad E_n^{(2)}(\alpha, \beta) = \frac{(i)^n}{(\pi)^{1/2}} \int_C e^{\alpha^2/\tau^2 - \tau^2/4} \chi_n\left(\beta, -\frac{\alpha^2}{\tau^2}\right) d\tau,$$

where the contour starts at the origin in the direction of the positive imaginary axis, bends around and extends to infinity along the positive real axis; and where $\chi_n(s, t)$ is a polynomial related to the Hermite polynomial,†

$$(21) \qquad \begin{aligned} \chi_n(s, t) &= \frac{(2t)^n}{n!} e^{-s^2/(4t)} \frac{\partial^n}{\partial s^n} \left(e^{s^2/(4t)}\right) = s^n \sum_{m=0}^{\leq n/2} \sigma_{n,m} \left(\frac{t}{s^2}\right)^m \\ &= \frac{(-t)^{n/2}}{n!} He_n\left(\frac{s}{2it^{1/2}}\right). \end{aligned}$$

A similar expression exists for $E^{(1)}(\alpha, \beta)$, where in (20) i is replaced by $-i$ and the contour is the same but below the positive real axis.

In conclusion it might be stated that on setting $\beta = 0$ the expansions (1)-(4) become simply special cases of Lommel's expansion of a Bessel function with an irrational variable.‡ An extension of these expansions to obtain generalizations of our functions of two variables might be of some interest, although it introduces hypergeometric series of the type ${}_2F_3$.

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* Watson, loc. cit., p. 178.

† H. Lacrose, Comptes Rendus, vol. 148 (1909), pp. 770–773.

‡ Watson, loc. cit., p. 140.

ON THE MODULUS OF THE DERIVATIVE OF A POLYNOMIAL*

BY W. E. SEWELL

1. *Introduction.* Let $P_n(z)$ be an arbitrary polynomial of degree n in z and let $|P_n(z)| \leqq M$ on a set C. The modulus of $P_n'(z)$† on C has an upper bound depending on M, on n, and on the set C. In this connection A. Markoff‡ has proved the following theorem.

Let $|P_n(z)| \leqq 1$ *in the interval* $-1 \leqq z \leqq +1$. *Then* $|P_n'(z)| \leqq n^2$ *for* $-1 \leqq z \leqq +1$. *This bound is attained only by the polynomial* $\pm\alpha \cos n \arccos z$, $|\alpha| = 1$.

A second fundamental result is the following theorem of S. Bernstein.§

Let $|P_n(z)| \leqq 1$ *on* C: $|z| \leqq 1$. *Then* $|P_n'(z)| \leqq n$ *on* C. *This bound is attained only by the polynomial* αz^n, $|\alpha| = 1$.

These theorems have been generalized in various directions by P. Montel,‖ G. Szegö,¶ Dunham Jackson,** and the author.†† Here we will prove the following generalization.

THEOREM A. *Let* $P_n(z)$, *a polynomial of degree* n *in* z, *be in modulus less than a constant* M *on a set* C *which has no isolated points and whose complement has finite connectivity. Then*

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* Presented to the Society, December 31, 1935.

† $P_n'(z)$ denotes the first derivative of $P_n(z)$.

‡ A. Markoff, Abhandlungen der Akademie der Wissenschaften zu St. Petersburg, vol. 62 (1889), pp. 1–24. Markoff considers only polynomials with real coefficients. For the general case see M. Riesz, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354–368; see especially p. 357.

§ S. Bernstein, *Leçons sur les Propriétés Extrêmales*, 1926, pp. 44–46.

‖ P. Montel, Bulletin de la Société Mathématique de France, vol. 46 (1919), pp. 151–196.

¶ G. Szegö, Mathematische Zeitschrift, vol. 23 (1925), pp. 45–61.

** Dunham Jackson, this Bulletin, vol. 36 (1930), pp. 851–857; vol. 37 (1931), pp. 883–890.

†† W. E. Sewell, Proceedings National Academy of Sciences, vol. 21 (1935), pp. 255–258.

$|P_n'(z)| \leqq MK(C)n^2$ *on* C, *where the constant* $K(C)$ *depends only on* C.

An application* of this result gives the following theorem.

THEOREM B. *Let* $f(z)$ *be a function of* z *defined on* C. *If for every* n *there exists a polynomial of degree* n, *such that*

$$|f(z) - P_n(z)| < \frac{M}{n^\alpha}, \qquad (\alpha > 2),$$

on C, α *and* M *independent of* n *and* z, *then* $f(z)$ *has a first derivative on* C.

2. *Proof of Theorem A.* The result is obvious if $P_n(z)$ is a constant and in all other cases the set C is bounded. The theorem is not true for an isolated point.† Let C^ν be any one of the finite number of pieces composing the set C; the complement D^ν of C^ν is simply connected and contains the point $z=\infty$. Let $w=\phi(z)$ map D^ν conformally on $|w|>1$ so that the points at infinity correspond. Call the image of $|w|=1+\rho$, $(\rho>0)$, under the inverse map, C_ρ^ν. By a result of the author‡ it is known that the distance from C^ν to C_ρ^ν is at least as great as $B(C^\nu)\rho^2$, where $B(C^\nu)$ is a constant depending only on C^ν. Also by Walsh's§ generalization of a theorem of Bernstein, it is true that $|P_n(z)| \leqq M(1+\rho)^n$, z on or within C_ρ^ν. Now let z_0 be any point on C^ν and describe a circle γ about z_0 of radius $B(C^\nu)\rho^2$. Then

$$P_n'(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{P_n(t)dt}{(t-z_0)^2}.$$

Since γ lies interior to C_ρ^ν, we know that $|P_n(t)| \leqq M(1+\rho)^n$, for t on γ, and consequently

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* Similar results for generalized derivatives in the case of functions of a real variable have been established by P. Montel, loc. cit. Montel's results have been extended to the complex domain for various types of regions by the author, loc. cit. For applications of Bernstein's and Markoff's theorems to approximation in the sense of least mth powers see Dunham Jackson, loc. cit.

† Consider a polynomial at the origin, for example.

‡ Loc. cit., p. 257. Here the result is stated without proof.

§ J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, Colloquium Publications of this Society, vol. 20, 1935, pp. 77–78.

$$|P_n'(z_0)| \leqq \frac{1}{2\pi} \frac{M(1+\rho)^n 2\pi B(C_\nu)\rho^2}{[B(C_\nu)\rho^2]^2} = \frac{M}{B(C_\nu)} \frac{(1+\rho)^n}{\rho^2}.$$

If we take* $\rho = 2/(n-2)$, $(n>2)$, we have

$$|P_n'(z_0)| \leqq MK(C_\nu)n^2, \qquad (n>2),$$

where $K(C_\nu)$ is independent of n for $n>2$. Since we may consider a polynomial of degree $n-1$ as a polynomial of degree n with leading coefficient 0, the constant can be adjusted so that the inequality holds for all n.† There are only a finite number of pieces C_ν, and the proof is complete.

3. *Proof of Theorem B.* By hypothesis

$$|f(z) - P_n(z)| < \frac{M}{n^\alpha} \qquad \text{on } C,$$

$$|f(z) - P_{n+1}(z)| < \frac{M}{(n+1)^\alpha} \text{ on } C;$$

then

$$|P_{n+1}(z) - P_n(z)| < \frac{2M}{n^\alpha} \qquad \text{on } C.$$

Hence by Theorem A

$$|P_{n+1}'(z) - P_n'(z)| < \frac{2MK(C)(n+1)^2}{n^\alpha} \text{ on } C.$$

By taking $2^{m-1} \leqq n < 2^m$ and considering the remainder of the series‡

$$[P_{2^m}'(z) - P_n'(z)] + [P_{2^{m+1}}'(z) - P_{2^m}'(z)] + \cdots,$$

we can establish uniform convergence of the series

$$P_0'(z) + [P_1'(z) - P_0'(z)] + [P_2'(z) - P_1'(z)] + \cdots$$

for any $\alpha > 2$, and thus the proof is complete.

HARVARD UNIVERSITY

* For the details of this method see P. Montel, loc. cit., and G. Szegö, loc. cit.

† J. L. Walsh suggested this method for including all n.

‡ For the details of this method see P. Montel, loc. cit.

ON CERTAIN THEOREMS OF PÓLYA AND BERNSTEIN

BY NORMAN LEVINSON*

1. *Introduction.* As a generalization of the Fabry gap theorem Pólya† has proved the following result.

THEOREM 1. *The function $f(z)$ given by the power series*

$$f(z) = \sum_0^\infty a_n z^{m_n} \tag{1}$$

with

$$\lim_{n\to\infty} \frac{n}{m_n} = D, \tag{2}$$

has‡ at least one singularity on every arc on its circle of convergence whose intercepted angle exceeds $2\pi D$.

V. Bernstein§ has proved the following theorem.

THEOREM 2. *Let $\phi(z)$ be analytic in some sector $|\operatorname{am} z| \leqq \alpha$. Let*

$$\overline{\lim_{r\to\infty}} \frac{\log|\phi(r)|}{r} = a. \tag{3}$$

Suppose there exists some b such that for any $\epsilon > 0$,

$$\phi(re^{i\theta}) = O(e^{(a\cos\theta + b|\sin\theta| + \epsilon)r}), \qquad (|\theta| \leqq \alpha). \tag{4}$$

Let $\{\lambda_n\}$ be an increasing sequence of positive numbers such that

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = D, \qquad \lambda_{n+1} - \lambda_n \geqq d > 0. \tag{5}$$

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* National Research Fellow.

† Pólya, *Untersuchungen über Lücken und Singularitäten von Potenzreihen*, Mathematische Zeitschrift, vol. 29 (1929).

‡ The theorem with (2) replaced by the Pólya maximal density is obviously an immediate corollary of Theorem 1.

§ Bernstein, *Séries de Dirichlet*, 1933, Chap. 9.

If $\pi D > b$, then

$$\overline{\lim_{n\to\infty}} \frac{\log |\phi(\lambda_n)|}{\lambda_n} = a = \overline{\lim_{r\to\infty}} \frac{\log |\phi(r)|}{r}. \tag{6}$$

Bernstein uses rather deep results from the theory of Dirichlet series in proving this theorem and points out that as yet no proof has been given using ordinary function theory. Here we will give a simple function theoretic proof of this theorem and then show how it leads at once to a proof of Theorem 1. We will then discuss other theorems of Pólya and Bernstein.

2. *Proofs of Theorems* 1 *and* 2. We require the following results of Pólya* and Carlson.

Let $\{\lambda_n\}$ satisfy (5). *If*

$$F(z) = \prod_1^\infty \left(1 - \frac{z^2}{\lambda_n^2}\right),$$

then

$$\lim_{r\to\infty} \frac{\log |F(re^{i\theta})|}{r} = \pi D \sin |\theta|, \qquad (\theta \neq 0, \pi).$$

$$\overline{\lim_{x\to\infty}} \frac{\log |F(\pm x)|}{x} = 0. \tag{7}$$

To any $\epsilon > 0$ there exists an A and a B such that

$$|F(z)| > Ae^{-\epsilon|z|}, \qquad |z \pm \lambda_n| \geqq \frac{1}{2} d, \tag{8}$$

and

$$|F'(\lambda_n)| > Be^{-\epsilon\lambda_n}. \tag{9}$$

PROOF OF THEOREM 2. Let us suppose that $\pi D > b$ but that contrary to (6) we have

$$\overline{\lim_{n\to\infty}} \frac{\log |\phi(\lambda_n)|}{\lambda_n} < c < a. \tag{10}$$

From (9) and (10)

* For a proof of these results see Pólya, loc. cit., p. 568, Theorem XI and the two examples following it.

$$g(z) = \sum_1^\infty \frac{\phi(\lambda_n)e^{-c\lambda_n}}{F'(\lambda_n)(z-\lambda_n)} e^{cz}F(z) \tag{11}$$

is an entire function. Since $g(\lambda_n) = \phi(\lambda_n)$, it follows that

$$\psi(z) = \frac{\phi(z) - g(z)}{F(z)} \tag{12}$$

is analytic for $|\text{am } z| \leqq \alpha$. From (4), (8), and (11), it follows that there exists a $B>0$ such that, for $|z-\lambda_n| \geqq d/2$,

$$\psi(z) = O(e^{B|z|}). \tag{13}$$

Since $\psi(z)$ is analytic (13) must be true even when $|z-\lambda_n| < d/2$.

We assume $\alpha < \pi/2$, which is obviously no restriction. If we set $(\pi D - b)\tan\alpha = \gamma$, then (4), (7), (11), and (12) give us

$$\psi(re^{\pm i\alpha}) = O(e^{r\cos\alpha(a-\gamma+\epsilon\sec\alpha)} + e^{cr\cos\alpha}).$$

If we now choose $\epsilon < \gamma\cos\alpha/2$, it is clear that

$$\psi(re^{\pm i\alpha}) = O(e^{rp\cos\alpha}), \qquad p = \max(a - \gamma/2, c).$$

That is, $e^{-pz}\psi(z)$ is bounded on $\text{am } z = \pm\alpha$. But by a well known theorem of Phragmén-Lindelöf this and (13) imply that it is bounded in the entire sector $|\text{am } z| \leqq \alpha < \pi/2$. Thus, in particular,

$$\psi(x) = O(e^{px}), \qquad p = \max(a - \gamma/2, c). \tag{14}$$

By (12), $\phi(x) = F(x)\psi(x) + g(x)$. Thus using (7), (11), and (14), we find

$$\overline{\lim_{x\to\infty}} \frac{\log|\phi(x)|}{x} \leqq \max(a - \gamma/2, c),$$

contrary to (3). This proves the theorem.

PROOF OF THEOREM 1. We suppose without restriction that the radius of convergence is 1. Suppose there exists an arc of the unit circle with intercepted angle $2\pi\beta > 2\pi D$ on which $f(z)$ is analytic. We can assume that the arc contains the point $z=-1$ and is symmetric about this point. We define C as the curve consisting of the line from $(1+\delta)e^{-\pi i(1-\beta)}$ to $e^{-\pi i(1-\beta)}/2$, the arc of the circle $|z| = 1/2$ from $e^{-\pi i(1-\beta)}/2$ to $e^{\pi i(1-\beta)}/2$ in the positive sense, and the line from $e^{\pi i(1-\beta)}/2$ to $(1+\delta)e^{\pi i(1-\beta)}$. We define J

as the curve along the arc of the circle $|z| = 1+\delta$ from the point $(1+\delta)e^{\pi i(1-\beta)}$ to the point $(1+\delta)e^{-\pi i(1-\beta)}$ in the positive sense.

We take $\delta > 0$ but so small that $f(z)$ is analytic on the closed curve $C+J$ and inside the region bounded by this curve.

Let $\{\mu_n\}$ be the positive integers complementary to the $\{m_n\}$. Then clearly

$$\lim_{n\to\infty} \frac{n}{\mu_n} = 1 - D. \tag{15}$$

Since the coefficients of z^{μ_n} in the power series of $f(z)$ are zero,

$$\int_C \frac{f(z)}{z^{\mu_n+1}}\,dz + \int_J \frac{f(z)}{z^{\mu_n+1}}\,dz = 0. \tag{16}$$

Since $f(z)$ is bounded on J, (16) gives

$$\int_C \frac{f(z)}{z^{\mu_n+1}}\,dz = O\left(\frac{1}{(1+\delta)^{\mu_n}}\right). \tag{17}$$

Let

$$H(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{w+1}}\,dz. \tag{18}$$

Writing z^{-w} as $r^{-w}e^{-iw\theta}$, we see that

$$|z^{-u-iv}| \leqq r^{-u}e^{\pi|v|(1-\beta)}$$

on C. Therefore for $\delta < 1$,

$$H(u+iv) = O(2^{|u|}e^{\pi|v|(1-\beta)}). \tag{19}$$

Let

$$\lim_{u\to\infty} \frac{\log|H(u)|}{u} = a. \tag{20}$$

Then by (19),

$$H(w)e^{-(a+\epsilon)w+\pi i(1-\beta)w}$$

is bounded along the positive imaginary axis and by (20) with any $\epsilon > 0$, it is bounded along the positive real axis. Thus by a well known theorem of Phragmén and Lindelöf it is bounded in the first quadrant. That is, in the first quadrant

$$H(w) = O(e^{(\alpha+\epsilon)u+\pi(1-\beta)|v|}), \qquad (u \geqq 0). \tag{21}$$

Similarly this result is true in the fourth quadrant and therefore for all $u \geqq 0$. From (17) with $\delta < 1$,

$$H(\mu_n) = O((1+\delta)^{-\mu_n}) = O(e^{-\delta\mu_n/2}). \tag{22}$$

We can now apply Theorem 1 to $H(w)$ using (20), (21), (22), and (15). The fact that $\pi(1-D) > \pi(1-\beta)$ shows that $a \leqq -\delta/2$. That is,

$$H(u) = O(e^{-u\delta/4}), \qquad (u > 0).$$

Thus

$$a_n = H(\lambda_n) + \frac{1}{2\pi i}\int_J \frac{f(z)}{z^{\lambda_n+1}}\,dz$$

$$= H(\lambda_n) + O\left(\frac{1}{(1+\delta)^{\lambda_n}}\right) = O(e^{-\delta\lambda_n/4}).$$

Therefore the radius of convergence exceeds 1, contrary to our assumption. This proves the theorem.

3. *Extended Results.* We can get all the theorems of Bernstein of the same type as Theorem 2 which are given in Chapter 9 of his *Séries de Dirichlet* as easily as Theorem 2. Moreover, we can replace the real sequence $\{m_n\}$ by a complex sequence $\{z_n\}$, where $|\text{am } z_n| \to 0$ as $n \to \infty$. It is also possible to obtain certain new theorems of a similar nature but considerably more precise.

As regards Theorem 1, the method used will also yield the gap theorems of Pólya on entire functions. Here we require a more precise statement of Theorem 2 which is however no harder to prove. We will consider these various results in some detail in the near future.

PRINCETON UNIVERSITY

A PARADOX OF LEWIS'S STRICT IMPLICATION

BY TANG TSAO-CHEN

The postulates for Lewis's strict implication are nine in number,* namely,

[11.1] $pq \dashv qp$

[11.2] $pq \dashv p$

[11.3] $p \dashv pp$

[11.4] $(pq)r \dashv p(qr)$

[11.5] $p \dashv \sim(\sim p)$

[11.6] $p \dashv q.q \dashv r :\dashv .p \dashv r$

[11.7] $p.p \dashv q :\dashv .q$

[19.01] $\Diamond pq \dashv \Diamond p$

[20.01] $(\exists p, q) :\sim(p \dashv q).\sim(p \dashv \sim q).$

By the operations of substitution, adjunction, and inference, a body of theorems is obtained. But the following theorem, which is a paradox of the strict implication, is not explicitly mentioned in Lewis's book.

Any two of the first eight postulates are such that each is deducible from the other, if $p \dashv q$ be interpreted as 'p is deducible from q.'

In order to prove this theorem we assume the following eight theorems.†

1. $p \sim p = q \sim q$

Def. $0 = q \sim q$

* The references are to *Symbolic Logic*, by Lewis and Langford, 1932.

† For the proof of these theorems see the paper, *The theorem "$p \dashv q \cdot = \cdot pq = p$" and Huntington's relation between Lewis's strict implication and Boolean algebra*, by Tang Tsao-Chen in this Bulletin, vol. 42 (1936), pp. 743–746.

2. $p \sim p = 0$

3. $p0 = 0$

Def. $i = \sim \Diamond 0$

4. $pq \supset p. = .i$

5. $p \supset p. = .i$

6. $p \supset q. \supset .i$

7. $p \supset q. = :i.p \supset q$

8. $p \supset q. = .pq = p.$

Note that the Theorems 4 and 5 are particular cases of the following theorem.

9. *If $p \supset q$ is asserted, then $p \supset q \cdot = \cdot i$.*

[Hyp.] $p \supset q$ (1)

[(1), 8.] $pq = p$ (2)

[12.11] $pq = p. = .pq = p$ (3)

[(2), (3)] $pq = p. = .p = p$ (4)

[11.03, 12.7] $p = p. = .p \supset p$ (5)

[(4), (5), 5.] $pq = p. = .i$ (6)

[(6), 8.] $p \supset q. = .i$

From the above theorem it is very easy to prove the following theorem.

10. *If $p \supset q$ and $r \supset s$ are both asserted, then*

$$p \supset q. \supset .r \supset s \quad (1)$$

and

$$r \supset s. \supset .p \supset q. \quad (2)$$

[Hyp.] $p \supset q$ (3)

[(3), 9.] $p \supset q. = .i$ (4)

[Hyp.] $r \supset s$ (5)

[(5), 9.]	$r \supset s. = .i$	(6)
[(4), (6)]	$p \supset q. = .r \supset s$	(7)
[11.03]	(7) = (1)(2)	(8)
[(7), (8)]	(1)(2)	(9)
[11.2]	(1)(2) $\supset$ (1)	(10)
[12.17]	(1)(2) $\supset$ (2)	(11)
[(9), (10)]	(1)	
[(9), (11)]	(2) .	

The paradox stated above is a particular case of Theorem 10, and therefore requires no further proof.

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THE BETTI NUMBERS OF CYCLIC PRODUCTS

BY R. J. WALKER

1. *Introduction.* In a recent paper† M. Richardson has discussed the symmetric product of a simplicial complex and has obtained explicit formulas for the Betti numbers of the two- and three-fold products. Acting on a suggestion of Lefschetz, we define a more general type of topological product and apply Richardson's methods to compute the Betti numbers of a certain one of these, the "cyclic" product.

2. *Basis for m-Cycles of General Products.* Let S be a topological space and G a group of permutations on the numbers $1, \cdots, n$. The *product of S with respect to G, $G(S)$*, is the set of all n-tuples $(P_1, \cdots, P_n)$ of points of S, where $(P_{i_1}, \cdots, P_{i_n})$ is to be regarded as identical with $(P_1, \cdots, P_n)$ if and only if the permutation $\left(\begin{smallmatrix} 1 & \cdots & n \\ i_1 & \cdots & i_n \end{smallmatrix}\right)$ is an element of G. A neighborhood of $(P_1, \cdots, P_n)$ is the set of all points $(Q_1, \cdots, Q_n)$ for which Q_i belongs to a fixed neighborhood of P_i. It is not difficult to verify that the

† M. Richardson, *On the homology characters of symmetric products*, Duke Mathematical Journal, vol. 1 (1935), pp. 50–69. We shall refer to this paper as R.

Hausdorff axioms hold for this definition of neighborhood, and hence that $G(S)$ is a topological space. In particular, if G is the identity or the symmetric group, $G(S)$ is, respectively, the direct or the symmetric product of S. If G is the cyclic group on n elements we shall call $G(S)$ the n-fold *cyclic* product of S.

The space $G(S)$ can be obtained in another manner. Let S^n denote the n-fold direct product of S. Then each element $\binom{1\cdots n}{i_1\cdots i_n}$ of G gives rise to an automorphism of S^n which carries $(P_1, \cdots, P_n)$ into $(P_{i_1}, \cdots, P_{i_n})$. By identifying points which are images of each other under the group of automorphisms we evidently obtain a space homeomorphic to $G(S)$.

Now let K be a simplicial complex, K^n its direct product, and $k=G(K)$ its product with respect to the group G of degree n and order r. We then have r automorphisms T_λ of K^n, and a continuous, single-valued transformation Λ of K^n into k, such that†

$$\Lambda T_\lambda = \Lambda. \tag{1}$$

Richardson has shown‡ that K^n and k can be subdivided into simplexes in such a fashion that the transformations T_λ and Λ are simplicial. We can therefore operate with them on chains of K^n. If E and e are simplexes of K^n and k, respectively, such that $e=\Lambda E$, we define the operator Λ' by $\Lambda' e=\sum_\lambda T_\lambda E$. We have then

$$\Lambda\Lambda' e = re, \tag{2}$$

$$\Lambda'\Lambda E = \sum_\lambda T_\lambda E. \tag{3}$$

We also find that T_λ, Λ, and Λ' preserve boundaries and hence homologies.

The principal theorem of Richardson,§ concerning the Betti numbers of k, is stated in terms of matrices. For actual computation we find it easier to work with the cycles themselves, and so we shall state and prove the theorem in a slightly different form.

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† In the expression for the product of two transformations, the transformation represented by the right-hand symbol is to be applied first.

‡ R, pp. 51 and 53.

§ R, p. 52.

THEOREM 1. *Let* $\{\Gamma^i\}$ *be an independent basis, with respect to homology, for m-cycles, with rational coefficients, of* K^n, *such that* $T_\lambda\Gamma^i = \pm\Gamma^{i_\lambda}$, $(\lambda = 1, \cdots, r)$; *and let* $\{\overline{\Gamma}^\alpha\}$ *be a maximal subset of* $\{\Gamma^i\}$ *such that*

(a) $$T_\lambda\overline{\Gamma}^\alpha \neq \pm\overline{\Gamma}^\beta, \qquad (\alpha \neq \beta),$$

(b) $$T_\lambda\overline{\Gamma}^\alpha \neq -\overline{\Gamma}^\alpha,$$

for any λ. *Then* $\{\gamma^\alpha\} = \{\Delta\overline{\Gamma}^\alpha\}$ *is an independent basis with respect to homology for the m-cycles of k.*

PROOF. (i) The γ^α are independent. For suppose that we have $\sum_\alpha x_\alpha\gamma^\alpha \sim 0$, that is, $\sum_\alpha x_\alpha\Delta\overline{\Gamma}^\alpha \sim 0$. Then

$$\Delta'\sum_\alpha x_\alpha\Delta\overline{\Gamma}^\alpha = \sum_\alpha x_\alpha\Delta'\Delta\overline{\Gamma}^\alpha = \sum_{\alpha,\lambda} x_\alpha T_\lambda\overline{\Gamma}^\alpha \sim 0,$$

by (3). Now if $T_\lambda\overline{\Gamma}^\alpha = \epsilon\Gamma^i$, $\epsilon = \pm 1$, we cannot have $T_\mu\overline{\Gamma}^\alpha = -\epsilon\Gamma^i$, for this would imply

$$T_\mu^{-1}T_\lambda\overline{\Gamma}^\alpha = \epsilon T_\mu^{-1}\Gamma^i = -\epsilon^2\overline{\Gamma}^\alpha = -\overline{\Gamma}^\alpha,$$

contrary to condition (b). Similarly, from (a), we cannot have $T_\mu\overline{\Gamma}^\beta = \pm\Gamma^i$, $\beta \neq \alpha$. Hence with each such Γ^i there is associated an ϵ_i, a $\overline{\Gamma}^\alpha$, and s_i values of λ for which $T_\lambda\overline{\Gamma}^\alpha = \epsilon_i\Gamma^i$. If the last homology is now written in terms of the basis $\{\Gamma^i\}$, the coefficient of Γ^i will be $\epsilon_i s_i x_\alpha$. Since the Γ^i are independent, $\epsilon_i s_i x_\alpha = 0$, and therefore every $x_\alpha = 0$.

Use was made of the properties of the rational coefficients only in the last step of each part of the proof. Now the s_i introduced in (i) are factors of r, for the T_λ for which $T_\lambda\overline{\Gamma}^\alpha = \epsilon_i\Gamma^i$ evidently form a coset of the subgroup which leaves $\overline{\Gamma}^\alpha$ invariant. It follows that the theorem will hold for any coefficient group in which each element has a unique rth part; in particular for the group of residues modulo a number prime to r.

(ii) $\{\gamma^\alpha\}$ is a basis. We note first that since the set $\{\overline{\Gamma}^\alpha\}$ is maximal every Γ^i is of one of the two forms $T_\lambda\overline{\Gamma}^\alpha$ or $\overline{\Gamma}^j$, where for each j there is a λ_j such that $T_{\lambda_j}\overline{\Gamma}^j = -\overline{\Gamma}^j$. Also, $\Delta\overline{\Gamma}^j = \Delta T_{\lambda_j}\overline{\Gamma}^j = -\Delta\overline{\Gamma}^j$, so that $\Delta\overline{\Gamma}^j = 0$. Now if γ is any m-cycle of k, $\Delta'\gamma$ is an m-cycle of K^n, and so

$$\Delta'\gamma \sim \sum_i x_i\Gamma^i = \sum_{\alpha,\lambda} x_{\alpha\lambda}T_\lambda\overline{\Gamma}^\alpha + \sum_j x_j\overline{\Gamma}^j.$$

Hence

$$\Lambda\Lambda'\gamma = r\gamma \sim \sum_{\alpha,\lambda} x_{\alpha\lambda}\Lambda T_\lambda \Gamma^\alpha + \sum_j x_j \Lambda \tilde{\Gamma}^j = \sum_{\alpha,\lambda} x_{\alpha\lambda}\gamma^\alpha,$$

by (2) and (1). That is,

$$\gamma \sim \sum_{\alpha,\lambda} \frac{x_{\alpha\lambda}}{r}\gamma^\alpha,$$

3. *Betti Numbers of Cyclic Products.* Keeping the notation as before, we let G be the cyclic group on n elements. To compute the mth Betti number of the cyclic product k we must count the number of m-cycles Γ^α. A basis of the type $\{\Gamma^i\}$ used in the theorem is obtained by taking all cycles of the form

$$C_{m_1} \times \cdots \times C_{m_n}, \qquad m_1 + \cdots + m_n = m,$$

C_{m_i} being a member of a basis of m_i-cycles of K.† Following Richardson's procedure, we obtain

$$T_\lambda(C_{m_1} \times \cdots \times C_{m_\lambda} \times C_{m_{\lambda+1}} \times \cdots \times C_{m_n}) = (-1)^{\epsilon_\lambda} C_{m_{\lambda+1}} \times \cdots \times C_{m_n} \times C_{m_1} \times \cdots \times C_{m_\lambda},$$

where

$$\epsilon_1 = m_1m_2 + \cdots + m_1m_n = m_1(m - m_1) = mm_1 - m_1^2 \equiv mm_1 - m_1 \pmod 2 = (m-1)m_1,$$

and by induction

$$\epsilon_\lambda \equiv (m-1)(m_1 + \cdots + m_\lambda) \pmod 2.$$

Let q be a factor of n, $n = qs$, and consider all Γ^i which are invariant, to within change of sign, under G_q, the cyclic subgroup of G of order q. They necessarily have the form

$$\Gamma_q = (C_{m_1} \times \cdots \times C_{m_s}) \times (C_{m_1} \times \cdots \times C_{m_s}) \times \cdots \times (C_{m_1} \times \cdots \times C_{m_s}),$$

there being q identical sets of factors. We must have $q(m_1 +$

† S. Lefschetz, *Topology*, p. 228.

$\cdots + m_s) = m$; that is, to have a Γ_q, q must be a factor of m and hence of (m, n), the highest common factor of m and n. If t is a proper multiple of q and a factor of (m, n), it is easily seen that a Γ_t is also a Γ_q. We denote by Γ_q^* any Γ_q which is not such a Γ_t, and by $A_{m,q}$ the number of Γ_q^*. The total number of Γ_q is then $\sum_t A_{m,t}$, the summation being over all values of t which are multiples of q and factors of (m, n). But the number of Γ_q is evidently equal to the number of possible combinations of the form $C_{m_1} \times \cdots \times C_{m_s}$, $m_1 + \cdots + m_s = m/q$, and this is exactly $R_{m/q}(K^s)$. Hence

$$\sum_t A_{m,t} = R_{m/q}(K^{s/q}),$$

and from these equations we can obtain the $A_{m,q}$ step by step starting with $q = (m, n)$, or directly by the use of the Dedekind inversion formula.

Now

$$T_s\Gamma_q = (-1)^{(m-1)(m_1+\cdots+m_s)}\Gamma_q = (-1)^{(m-1)m/q}\Gamma_q,$$

and so if m is even and m/q is odd, Γ_q is a cycle of the type $\bar{\Gamma}^j$ of Theorem 1 and is not counted among the $\bar{\Gamma}^\alpha$. We therefore put

$$B_{m,q} = \begin{cases} 0, & \text{if } m \text{ is even and } m/q \text{ is odd,} \\ A_{m,q} & \text{otherwise.} \end{cases}$$

Consider the s cycles Γ_q^*, $T_1\Gamma_q^*$, $\cdots$, $T_{s-1}\Gamma_q^*$. If any two of these are equal, say $T_i\Gamma_q^* = T_j\Gamma_q^*$, $(i > j)$, then Γ_q^* is invariant, to within change of sign, under the subgroup generated by $T_j^{-1}T_i = T_{i-j}$, and hence under the minimal subgroup containing G_q and T_{i-j}. Since $i - j < s$, T_{i-j} is not an element of G_q and therefore this subgroup is a G_t with t a proper multiple of q, contrary to the definition of Γ_q^*. It follows that there are exactly $s = n/q$ distinct transforms of each of the $B_{m,q}$ cycles Γ_q^*, and so we can pick out $(q/n)B_{m,q}$ of the Γ_q^* which are not transformable into one another and which can therefore be included among the $\bar{\Gamma}^\alpha$ of Theorem 1. Since the cycles Γ_q^* for different values of q are not transformable into one another and since every Γ^i is a Γ_q^* for some q, we have the following result.

THEOREM 2.

$$R_m(k) = (1/n)\sum_q qB_{m,q},$$

the summation being over all factors of (m, n).

The following special cases may be of interest.

COROLLARY 1. *If* n *is an odd prime*

$$R_m(k) = \begin{cases} (1/n)R_m(K^n), & \text{if } (m, n) = 1, \\ (1/n)[R_m(K^n) - R_s(K)] + R_s(K), & \text{if } m = ns. \end{cases}$$

COROLLARY 2. *If* p *is an odd prime and* $n = p^\alpha$, $m = p^\beta m_1$, $(m_1, p) = 1$, *and* $\gamma = \min \alpha, \beta$,

$$R_m(k) = \frac{p-1}{n}\left[\frac{1}{p-1}R_m(K^n) + \sum_{i=1}^{\gamma} p^{i-1}R_{m/p^i}(K^{n/p^i})\right].$$

COROLLARY 3. *If* $R_0(K) = 1$, *then* $R_1(k) = R_1(K)$.

4. *Remark.* The methods used on the cyclic product can evidently be used to compute the Betti numbers of a product with respect to an arbitrary group. In general, however, the resulting formulas are too complicated to be of interest.

CORNELL UNIVERSITY

THE DIVERGENCE OF SEQUENCES OF POLYNOMIALS INTERPOLATING IN ROOTS OF UNITY

BY J. L. WALSH

Polynomials defined by the property of interpolation to a given function in the roots of unity have been studied by Méray, Runge, Fejér, and the present writer,* and are of considerable interest. Thus the following theorem has been established, except for the last sentence.

THEOREM. *Let the function $f(z)$ be single-valued and analytic throughout the interior of the circle $|z|=\rho>1$, but not single-valued and analytic throughout the interior of any larger concentric circle. Let $p_n(z)$ be the unique polynomial† of degree n which interpolates to $f(z)$ in all the $(n+1)$st roots of unity. Then the sequence $p_n(z)$ converges to $f(z)$ throughout the region $|z|<\rho$, uniformly on any closed point set interior to that region. The sequence $p_n(z)$ diverges at every point exterior to the circle $|z|=\rho$.*

Hitherto it has been known‡ merely that the sequence $p_n(z)$ diverges at every point of the region $\rho<|z|<\rho^2$; it is the object of the present note to establish divergence at every point exterior to the circle $|z|=\rho$.

In our proof it is convenient to have for reference several easily proved lemmas.

LEMMA 1. *The relations*

$$\overline{\lim_{n\to\infty}}\,|A_n|^{1/n}=q, \qquad \overline{\lim_{n\to\infty}}\,|B_n|^{1/n}=q_1\leqq q \tag{1}$$

imply

$$\overline{\lim_{n\to\infty}}\,|A_n+B_n|^{1/n}\leqq q. \tag{2}$$

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* See, for instance, Walsh, *Interpolation and Approximation*, Colloquium Publications of this Society, vol. 20, 1935, §§7.1, 7.10.

† A polynomial of degree n is an expression of the form $a_0z^n+a_1z^{n-1}+\cdots+a_n$: we do not assume $a_0\neq 0$.

‡ Walsh, op. cit., p. 154.

Let $\epsilon>0$ be arbitrary. From equations (1) we have for n sufficiently large

$$|A_n| \leqq (q+\epsilon)^n, \qquad |B_n| \leqq (q_1+\epsilon)^n \leqq (q+\epsilon)^n.$$

Thus we have

$$|A_n + B_n| \leqq 2(q+\epsilon)^n, \qquad \overline{\lim_{n\to\infty}} |A_n + B_n|^{1/n} \leqq q+\epsilon.$$

When ϵ is allowed to approach zero, inequality (2) follows.

LEMMA 2. *The relations*

$$\overline{\lim_{n\to\infty}} |C_n|^{1/n} = q, \qquad \overline{\lim_{n\to\infty}} |D_n|^{1/n} = q_1 < q \tag{3}$$

imply

$$\overline{\lim_{n\to\infty}} |C_n + D_n|^{1/n} = q. \tag{4}$$

From Lemma 1 we conclude that the left-hand member of (4) is not greater than q. But

$$\overline{\lim_{n\to\infty}} |C_n + D_n|^{1/n} = q_2 < q \tag{5}$$

is impossible, for by (5) and the second of relations (3) we should have by setting $A_n = C_n + D_n$, $B_n = -D_n$ in Lemma 1

$$\overline{\lim_{n\to\infty}} |C_n|^{1/n} \leqq [\text{larger of } q_1 \text{ and } q_2] < q,$$

in contradiction to (3).

LEMMA 3. *Let $s_n(z)$ denote the sum of the first $n+1$ terms of the series $\sum_{n=0}^{\infty} a_n z^n$. Let us suppose*

$$\overline{\lim_{n\to\infty}} |a_n|^{1/n} = A > 0. \tag{6}$$

Then we have for $|z| \geqq 1/A$

$$\overline{\lim_{n\to\infty}} |s_n(z)|^{1/n} = A|z|. \tag{7}$$

Let $\epsilon>0$ be arbitrary. For n sufficiently large we have $|a_n| < (A+\epsilon)^n$, so the sequence $|a_n|/(A+\epsilon)^n$ has a finite upper bound M:

$$|a_n| \leqq M(A+\epsilon)^n,$$

$$|s_N(z)| \leqq M\sum_{n=0}^{N}(A+\epsilon)^n|z|^n = M\frac{(A+\epsilon)^{N+1}|z|^{N+1}-1}{(A+\epsilon)|z|-1},$$

$$\overline{\lim_{N\to\infty}}\,|s_N(z)|^{1/N} \leqq (A+\epsilon)|z|;$$

allowing ϵ to approach zero then yields the relation

$$\overline{\lim_{n\to\infty}}\,|s_n(z)|^{1/n} \leqq A|z|. \tag{8}$$

Let us now assume the inequality to hold in (8):

$$\overline{\lim_{n\to\infty}}\,|s_n(z)|^{1/n} < Q < A|z|;$$

we shall reach a contradiction. For n sufficiently large we have

$$|s_{n-1}(z)| < Q^{n-1}, \qquad |s_n(z)| < Q^n,$$

$$|s_n(z) - s_{n-1}(z)| = |a_n z^n| < Q^n(1+1/Q),$$

$$|a_n| < Q^n(1+1/Q)/|z|^n,$$

$$\overline{\lim_{n\to\infty}}\,|a_n|^{1/n} \leqq Q/|z| < A,$$

which contradicts (6). Equation (7) is established.

It is worth remarking that equation (7) may fail if the requirement $|z| \geqq 1/A$ is omitted, as is illustrated by the series $\sum_{n=0}^{\infty} z^n$.

We are now in a position to establish the theorem. For the polynomials $p_n(z)$ already defined we have the interpolation formula

$$f(z) - p_n(z) = \frac{1}{2\pi i}\int_{C_R}\frac{z^{n+1}-1}{t^{n+1}-1}\,\frac{f(t)dt}{t-z}, \qquad (z \text{ interior to } C_R), \tag{9}$$

where C_R denotes the circle $|z| = R$, $1 < R < \rho$. If $P_n(z)$ denotes the sum of the first $n+1$ terms of the Taylor development of $f(z)$ about the origin, then $P_n(z)$ is also the polynomial of degree n which interpolates to $f(z)$ in the origin counted of multiplicity $n+1$, so we have

$$f(z) - P_n(z) = \frac{1}{2\pi i}\int_{C_R}\frac{z^{n+1}}{t^{n+1}}\,\frac{f(t)dt}{t-z}, \qquad (z \text{ interior to } C_R). \tag{10}$$

Subtraction of (10) from (9) member by member yields the equation

$$(11) \qquad P_n(z) - p_n(z) = \frac{1}{2\pi i}\int_{C_R} \frac{z^{n+1} - t^{n+1}}{t^{n+1}(t^{n+1} - 1)} \frac{f(t)dt}{t - z}.$$

Equation (11) is thus established for z interior to C_R. But when the common factor $t-z$ is cancelled from numerator and denominator in the integrand in (11), it is seen that that integrand considered as a function of z has no singularity for any (finite) value of z; hence the right-hand member of (11) represents a function of z analytic for all finite values of z; the two members of (11) are equal for z interior to C_R, so equation (11) is valid for all finite values of z.

For any finite value of z exterior to $|z| = \rho$ we read off directly from (11)

$$\overline{\lim_{n\to\infty}} \, | P_n(z) - p_n(z) |^{1/n} \leqq |z| / R^2.$$

But R can be allowed to approach ρ, whence the inequality

$$(12) \qquad \overline{\lim_{n\to\infty}} \, | P_n(z) - p_n(z) |^{1/n} \leqq |z| / \rho^2.$$

The radius of convergence ρ of the Taylor expansion $f(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ satisfies the equation

$$\overline{\lim_{n\to\infty}} \, | a_n |^{1/n} = 1/\rho,$$

and Lemma 3 yields for $|z| > \rho$,

$$(13) \qquad \overline{\lim_{n\to\infty}} \, | P_n(z) |^{1/n} = |z| / \rho.$$

Relations (12) and (13) imply through Lemma 2 ($|z| > \rho > 1$)

$$\overline{\lim_{n\to\infty}} \, | p_n(z) |^{1/n} = |z| / \rho > 1.$$

Thus the sequence $p_n(z)$ cannot be bounded when $|z| > \rho$, hence cannot converge. The theorem is established.

This method of proof is somewhat similar to, but by no means identical with, a proof used in an analogous situation by Faber.* The present proof is valid, the reader may notice, if the

* Journal für Mathematik, vol. 150 (1920), pp. 79–106.

polynomial $p_n(z)$ is defined by interpolation to $f(z)$ not in the $(n+1)$st roots of unity but in the $(n+1)$st roots of an arbitrary number α_n, $|\alpha_n| \leqq 1$.

Lemma 3 extends at once to the case of series of form

$$\sum_{n=0}^{\infty} a_n q_n(z), \tag{14}$$

where $\lim_{n=\infty} |q_n(z)|^{1/n} = q(z)$ exists. For such series, as for the more restricted case of power series, regions of convergence and of divergence can be expressed readily* in terms of $q(z)$ and of $\overline{\lim}_{n=\infty} |a_n|^{1/n}$. Many comparison series of form (14) found by interpolation exist, for which (as for power series) $\overline{\lim}_{n=\infty} |a_n|^{1/n}$ can be determined at once from the singularities of the function expanded; such a comparison series may be a series of interpolation, and may be an expansion in polynomials which belong to a more or less arbitrary region.† Whenever such a comparison series is given, the method of proof of the theorem of the present note yields the divergence of suitable "neighboring" sequences of polynomials of interpolation. Thus the present method has a much wider application than the italicized theorem suggests, by the use no longer of power series but of more general comparison series.

But if an arbitrary sequence of polynomials of interpolation is given (for instance, converging to the defining function maximally on a given point set‡), it may be difficult to determine a suitable comparison series (14) of the kind required for application of the present method. Thus there still remain many sequences of polynomials of interpolation for which exact regions of uniform convergence are known but whose divergence at all points exterior to those regions has never been established; still other methods need to be devised for the study of such divergence.

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* See op. cit., §3.4, Theorem 5.

† For instance, op. cit., Chapter 7, especially §7.2, Corollary 4 to Theorem 2.

‡ See op. cit., Chapter 7.

NOTE ON THE GREATEST INTEGER FUNCTION

BY M. A. BASOCO

1. *Introduction.* In this note we wish to record certain finite sums involving the greatest integer function $E(x)$, which seem to be of some interest. Hermite* has shown that the generating function for $E(x)$ has the simple form,

$$\frac{x^b}{(1-x)(1-x^a)} = \sum_{(n)} E\left(\frac{n+a-b}{a}\right) x^n, \tag{1}$$

where a, b are positive integers. To him is due, likewise, the development

$$\frac{x^b}{(1-x)(1+x^a)} = \sum_{(n)} E_1\left(\frac{n+a-b}{2a}\right) x^n, \tag{2}$$

where

$$E_1(x) = E(2x) - 2E(x) = E\left(x+\frac{1}{2}\right) - E(x). \tag{3}$$

As indicated by Hermite, these developments used in conjunction with the expansions for certain theta quotients, yield results of interest in the theory of numbers. Of particular importance are his results expressing some of Kronecker's class number sums (which arise from certain theta constants of the third degree) in terms of $E(x)$.

In what follows we are concerned with the application of Hermite's device to the twelve Jacobian elliptic functions and to the logarithmic derivatives of the theta functions. The results implicit in these theta series are first elicited in a form which involves an arbitrary function, which, when specialized to be x^r, yields relations between the sum of the rth powers of the divisors of the integer n and sums of $E(x)$ for certain values of the argument. For the analytical form of the series here used we

* Hermite, Acta Mathematica, vol. 5, pp. 310–324; Oeuvres, vol. 4, pp. 151–159. See also E. T. Bell, *Algebraic Arithmetic*, Colloquium Publications of this Society, vol. 7, p. 138.

refer to any standard treatise,* the corresponding arithmetized form being readily derived from them.†

2. *The Functions* ϕ_j, σ_j, ρ_j. Denote by $F(x)$ any arbitrary function and let m range over the odd numbers $1, 3, 5, \cdots$, while n is unrestricted. We define the functions ϕ_j, σ_j, ρ_j as follows:

$$\phi_1(x, n) = \sum F(\tau x), \qquad n = t\tau. \tag{4}$$

$$\phi_2(x, m) = \sum (-1/\tau)F(tx), \qquad m = t\tau. \tag{5}$$

$$\phi_3(x, n) = \sum (-1/\tau)F(2tx), \qquad n = t\tau. \tag{6}$$

$$\sigma_1(x, n) = \sum (-1)^{\delta}F(2dx), \qquad n = d\delta. \tag{7}$$

$$\sigma_2(x, n) = \sum (-1/\tau)F(\tau x), \qquad n = d\tau. \tag{8}$$

$$\sigma_3(x, n) = -\sum (-1)^{d+\delta}F(2dx), \qquad n = d\delta. \tag{9}$$

$$\rho_1(x, n) = \sum F(2tx), \qquad n = t\tau. \tag{10}$$

$$\rho_2(x, n) = \sum (-1)^{d}F(2dx), \qquad n = d\delta. \tag{11}$$

$$\rho_3(x, n) = \sum F(2dx), \qquad n = d\delta. \tag{12}$$

In each of the preceding, the summations range over the divisors d, δ, t, τ of the fixed positive integers n, m, the integer τ always being odd, and $(-1|h) = (-1)^{(h-1)/2}$ if h is odd.

3. *Sums in Terms of* $E(x)$. The following sums are deduced from the expansions of the functions

sn cn dn
ns cs ds
nc sc dc
nd sd cd,

using Glaishier's notation for the Jacobian elliptic functions. In our formulas, n is a fixed arbitrary positive integer, μ ranges over $1, 3, 5, \cdots$, while k ranges over $1, 2, 3, 4, \cdots, n$.

$$\sum_{(\mu)} \phi_1(x, \mu) = \sum_{(\mu)} E\left(\frac{n+\mu}{2\mu}\right)F(\mu x), \tag{13}$$

* See, for example, Whittaker and Watson, *Modern Analysis*, 4th ed., pp. 510–512.

† See E. T. Bell, Messenger of Mathematics, March, 1924, pp. 169–172, §§4, 14.

$$\sum_{(\mu)} \phi_2(x, \mu) = \sum_{(\mu)} E_1\left(\frac{n+\mu}{4\mu}\right) F(\mu x). \tag{14}$$

In the left side of these expressions, μ ranges up to $(n-1)$ or n according as n is even or odd.

$$\sum_{(k)} \phi_3(x, k) = \sum_{(k)} E_1\left(\frac{n+k}{4k}\right) F(2kx), \tag{15}$$

$$\sum_{(k)} \phi_1(x, k) = \sum_{(\mu)} E\left(\frac{n}{\mu}\right) F(\mu x), \tag{16}$$

$$\sum_{(k)} (-1)^{k-1} \phi_1(x, k) = \sum_{(\mu)} E_1\left(\frac{n}{2\mu}\right) F(\mu x), \tag{17}$$

$$\sum_{(k)} (-1)^{k-1} \sigma_2(x, k) = \sum_{(\mu)} \left(\frac{-1}{\mu}\right) E_1\left(\frac{n}{2\mu}\right) F(\mu x), \tag{18}$$

$$\sum_{(s)} \sigma_1\left(x, \frac{n-s+1}{2}\right) = -\sum_{(k)} E_1\left(\frac{n}{4k}\right) F(2kx), \tag{19}$$

$$\sum_{(s)} \sigma_3\left(x, \frac{n-s+1}{2}\right) = \sum_{(k)} (-1)^k E_1\left(\frac{n}{4k}\right) F(2kx). \tag{20}$$

In (19) and (20) the index s ranges over such values of $s \leqq n$ as will make $n-s \equiv 1$, mod 2.

$$\sum_{(k)} \sigma_2(x, k) = \sum_{(\mu)} \left(\frac{-1}{\mu}\right) E\left(\frac{n}{\mu}\right) F(\mu x), \tag{21}$$

$$\sum_{(k)} (-1)^{k-1} \phi_3(x, k) = \sum_{(k)} (-1)^{k-1} E_1\left(\frac{n+k}{4k}\right) F(2kx), \tag{22}$$

$$\sum_{(s)} (-1)^{(n-s)/2} \phi_1(x, n-s+1) = \sum_{(\mu)} \left(\frac{-1}{\mu}\right) E_1\left(\frac{n+\mu}{4\mu}\right) F(\mu x), \tag{23}$$

$$\sum_{(s)} \sigma_2(x, n-s+1) = \sum_{(\mu)} \left(\frac{-1}{\mu}\right) E\left(\frac{n+\mu}{2\mu}\right) F(\mu x). \tag{24}$$

In (23) and (24) the index s ranges over such values of $s \leqq n$ as will make $n-s \equiv 0$, mod 2.

The next results are deduced from the expansions of the logarithmic derivatives of the four elliptic theta functions.

$$\sum_{(k)} \rho_1(x, k) = \sum_{(k)} E\left(\frac{n+k}{2k}\right) F(2kx), \tag{25}$$

$$\sum_{(k)} (-1)^{k-1} \rho_1(x, k) = \sum_{(k)} (-1)^{k-1} E\left(\frac{n+k}{2k}\right) F(2kx), \tag{26}$$

$$\sum_{(s)} \rho_2\left(x, \frac{n-s+1}{2}\right) = \sum_{(k)} (-1)^k E\left(\frac{n}{2k}\right) F(2kx), \tag{27}$$

$$\sum_{(s)} \rho_3\left(x, \frac{n-s+1}{2}\right) = \sum_{(k)} E\left(\frac{n}{2k}\right) F(2kx). \tag{28}$$

In (27) and (28) the index s ranges over such values of $s \leqq n$ as will make $n-s \equiv 1$, mod 2.

4. *Sums of Powers of Divisors of n.* If in the preceding formulas the function $F(x)$ is taken to be x^r, r a positive integer, we immediately deduce the following, where the notation is as follows:

$\zeta_r(n)$ = sum of the rth powers of the divisors of n.

$\zeta_r'(n)$ = sum of the rth powers of the odd divisors of n.

$\psi_r(n)$ = sum of the rth powers of the divisors of n whose conjugates are odd.

$\delta_r(n)$ = the excess of the sum of the rth powers of the divisors of n whose conjugates are of the form $4k+1$ over the like sum for those whose conjugates are of the form $4k+3$.

$\delta_r'(n)$ = the excess of the sum of the rth powers of the $(4k+1)$ divisors of n over the like sum of the $(4k+3)$ divisors of n.

$\alpha_r(n)$ = the excess of the sum of the rth powers of the divisors of n whose conjugates are even over the like sum for those whose conjugates are odd.

$\beta_r'(n)$ = the sum of the rth powers of the even divisors of n diminished by the like sum of the odd divisors of n.

$\beta_r(n) = \sum(-1)^{d+\delta} d^r$, the sum extending over all the divisors d, δ of n.

$$\sum_{(\mu)} \zeta_r(\mu) = \sum_{(\mu)} E\left(\frac{n+\mu}{2\mu}\right) \mu^r, \tag{29}$$

$$\sum_{(\mu)} \delta_r(\mu) = \sum_{(\mu)} E_1\left(\frac{n+\mu}{4\mu}\right) \mu^r. \tag{30}$$

On the left side of these, μ ranges up to $(n-1)$ or n according as n is even or odd.

$$\sum_{(k)} \delta_r(k) = \sum_{(k)} E_1\left(\frac{n+k}{4k}\right) k^r, \tag{31}$$

$$\sum_{(k)} \zeta_r'(k) = \sum_{(\mu)} E\left(\frac{n}{\mu}\right) \mu^r, \tag{32}$$

$$\sum_{(k)} (-1)^{k-1} \zeta_r'(k) = \sum_{(\mu)} E_1\left(\frac{n}{2\mu}\right) \mu^r, \tag{33}$$

$$\sum_{(k)} (-1)^{k-1} \delta_r(k) = \sum_{(\mu)} \left(\frac{-1}{\mu}\right) E_1\left(\frac{n}{2\mu}\right) \mu^r, \tag{34}$$

$$\sum_{(s)} \alpha_r\left(\frac{n-s+1}{2}\right) = -\sum_{(k)} E\left(\frac{n}{4k}\right) k^r, \tag{35}$$

$$\sum_{(s)} \beta_r\left(\frac{n-s+1}{2}\right) = \sum_{(k)} (-1)^{k-1} E_1\left(\frac{n}{4k}\right) k^r. \tag{36}$$

In (35) and (36) the index s ranges over such values of $s \leqq n$ as will make $n-s \equiv 1$, mod 2.

$$\sum_{(k)} \delta_r'(k) = \sum_{(\mu)} \left(\frac{-1}{\mu}\right) E\left(\frac{n}{\mu}\right) \mu^r, \tag{37}$$

$$\sum_{(k)} (-1)^{k-1} \delta_r(k) = \sum_{(k)} (-1)^{k-1} E_1\left(\frac{n+k}{4k}\right) k^r, \tag{38}$$

$$\sum_{(s)} (-1)^{(n-s)/2} \zeta_r'(n-s+1) = \sum_{(\mu)} \left(\frac{-1}{\mu}\right) E_1\left(\frac{n+\mu}{4\mu}\right) \mu^r, \tag{39}$$

$$\sum_{(s)} \delta_r'(n-s+1) = \sum_{(\mu)} \left(\frac{-1}{\mu}\right) E\left(\frac{n+\mu}{2\mu}\right) \mu^r. \tag{40}$$

In (39) and (40) the index s ranges over such values of $s \leqq n$ as will make $n-s \equiv 0$, mod 2.

$$\sum_{(k)} \psi_r(k) = \sum_{(k)} E\left(\frac{n+k}{2k}\right) k^r, \tag{41}$$

$$\sum_{(k)} (-1)^{k-1} \psi_r(k) = \sum_{(k)} (-1)^{k-1} E\left(\frac{n+k}{2k}\right) k^r, \tag{42}$$

$$\sum_{(s)} \beta_r'\left(\frac{n-s+1}{2}\right) = \sum_{(k)} (-1)^k E\left(\frac{n}{2k}\right) k^r, \tag{43}$$

$$\sum_{(s)} \zeta_r\left(\frac{n-s+1}{2}\right) = \sum_{(k)} E\left(\frac{n}{2k}\right) k^r. \tag{44}$$

In (43) and (44) the index s ranges over such values of $s \leqq n$ as will make $n-s \equiv 1$, mod 2.

5. *Conclusion.* If in the expansion for dn we put $x=0$, we obtain

$$\vartheta_3^2(0, q) - 1 = \sum_{n=1}^{\infty} q^n f(n) = 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}}, \tag{45}$$

where $f(n)$ denotes the number of solutions of the equation

$$x^2 + y^2 = n, \qquad x, y \gtreqless 0. \tag{46}$$

From this it follows easily that

$$f(1) + f(2) + f(3) + \cdots + f(n) = 4\sum_{(k)} E_1\left(\frac{n+k}{4k}\right), \tag{47}$$

so that

$$f(n) = 4\sum_{(k)} \left\{ E_1\left(\frac{n+k}{4k}\right) - E_1\left(\frac{n+k-1}{4k}\right) \right\}. \tag{48}$$

Hermite, in a similar manner, has shown that

$$f(1) + f(2) + f(3) + \cdots + f(n) = 4\sum_{(\mu)} \left(\frac{-1}{\mu}\right) E\left(\frac{n}{\mu}\right). \tag{49}$$

Hence, we have

$$\sum_{(k)} E_1\left(\frac{n+k}{4k}\right) = \sum_{(\mu)} \left(\frac{-1}{\mu}\right) E\left(\frac{n}{\mu}\right). \tag{50}$$

In a similar manner,

$$\begin{aligned} \vartheta_2^2(0, q^4) = \sum q^n g(n) &= 4\sum_{(\mu)} \frac{q^{2\mu}}{1+q^{4\mu}} \\ &= 4\sum_{(\mu)} \left(\frac{-1}{\mu}\right) \frac{q^{2\mu}}{1-q^{4\mu}}, \end{aligned} \tag{51}$$

where $g(n)$ denotes the number of solutions of the equation (46) with x, y odd and $\gtrless 0$. Noting that $g(n)$ is zero unless $n \equiv 2$, mod 8, we obtain

$$g(2) + g(10) + \cdots + g(8r+2) = 4\sum_{(k)} E_1\left(\frac{2r+k}{4k-2}\right) = 4\sum (-1)^{k-1} E\left(\frac{2r+k}{2k-1}\right),$$

from which $g(8r+2)$ could be obtained explicitly in terms of either $E(x)$ or $E_1(x)$.

Lastly, since

$$\vartheta_0^2(0, q) - 1 = 4\sum_{(n)}^{\infty} (-1)^n \frac{q^n}{1+q^{2n}} = -4\sum_{(\mu)} \left(\frac{-1}{\mu}\right) \frac{q^\mu}{1+q^\mu}, \tag{53}$$

we obtain,

$$h(1) + h(2) + h(3) + \cdots + h(n) = 4\sum_{(\mu)} \left(\frac{-1}{\mu}\right) E_1\left(\frac{n}{2\mu}\right) = 4\sum_{(k)} (-1)^{k-1} E_1\left(\frac{n+k}{4k}\right), \tag{54}$$

where $h(n)$ denotes the excess of the number of solutions of (46) with x, y of opposite parity over the like with x, y of the same parity.

From (54) it follows that

$$h(n) = 4\sum_{(\mu)} \left(\frac{-1}{\mu}\right) \left\{ E_1\left(\frac{n}{2\mu}\right) - E_1\left(\frac{n-1}{2\mu}\right) \right\}. \tag{55}$$

Finally, it may be noted that a great many similar results are implicit in the expansions of the functions $\mathrm{sn}^k u$, $\mathrm{cn}^k u$, $\cdots$. The complete set of developments for the cases $k=3$, 4 have been given by the writer in a former paper.*

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* Tôhoku Mathematical Journal, vol. 35 (1932), pp. 35–42.

CERTAIN NON-INVOLUTORIAL CREMONA TRANSFORMATIONS OF HYPERSPACE

BY HARRIET F. MONTAGUE

1. *Introduction.* The problem here presented was suggested by a paper by Maria Miglio.* A four page synthetic outline given there for S_4 has been enlarged upon and extended to S_r in the present paper.

2. *Definition of the Transformation.* A non-involutorial Cremona transformation in S_r is set up as follows. A (1, 1) correspondence is established between the elements of a pencil of primals on a base o_1 and the points of a rational curve f_n of order n. A (1, 1) correspondence is also established between the elements of a pencil of primals on a base o_2 and the points of f_n. Any point P in S_r determines with o_1 an element of the first pencil and associated with this element there is a point A on f_n. Associated with A there is an element of the second pencil on o_2. The intersection of this element with the line AP gives P', the image of P in the transformation. The inverse transformation proceeds in a similar way starting with the pencil of elements on o_2.

This transformation is examined as applied to the following cases:

Case A. Pencil of primes on o_1, pencil of primes on o_2.

Case B. (a) Pencil of quadric primals on o_1, pencil of primes on o_2.

(b) Pencil of quadric primals on o_1, pencil of quadric primals on o_2.

Case C. (a) Pencil of cubic primals on o_1, pencil of primes on o_2.

(b) Pencil of cubic primals on o_1, pencil of quadric primals on o_2.

(c) Pencil of cubic primals on o_1, pencil of cubic primals on o_2.

Case D. Pencils of primals of higher order with f_n a line f_1.

* Maria Miglio, *Nell' S_4 una classe di trasformazioni birazionali*, Catania Accademia Gioenia Atti, (5), vol. 18 (1932) [Mem. 19], pp. 1–22.

3. *Case* A. The transformation is a T_{n+2} and the inverse is of the same order. The locus of invariant points is the locus of intersection of corresponding elements of the two pencils, as is found in all the cases following. The fundamental elements of the transformation are:

(1) The $n+1$ points of f_n through which pass the associated primes on o_1. Their images are the corresponding primes on o_2.

(2) The $n+1$ intersections of primes on o_2 which pass through their associated points on f_n with the corresponding primes on o_1. Their images are the $n+1$ primes on o_2 which pass through their associated points.

(3) The base o_1. Its image is the projection of a ruled surface of order $n+1$, containing o_1 once, o_2 n times, from the common part of o_1 and o_2 as vertex.

The residual intersection of two general homaloids, apart from the fundamental elements, is of order $n+2$. In S_3, C_{n+2} has $n+1$ points on o_1, one point on each of the simple fundamental lines, and $n+1$ points on f_n.

4. *Case* B. In using a pencil of quadric primals, the point associated with a particular quadric must lie on that quadric. If we have f_n a part of the base of the pencil of quadrics, in S_3, $n=1, 2,$ or 3; in S_r, $r>3$, f_n can be of any order.

(a) A pencil of quadric primals and a pencil of primes gives T_{2n+3} with the inverse T_{n+3}^{-1}.

The image of the base of the pencil of quadrics, a manifold M_4, is a primal. If M_4 does not contain f_n, M_4 meets f_n in $2n-1$ points. In S_3, the image of C_4 is a surface of order $2n+5$, containing o_2 to multiplicity $2n+1$. If f_n is a line f_1, f_1 counts once in the image of C_4. On the other hand, if f_1 is a part of the base, the image of the residual C_3 contains o_2 once and f_1 twice.

Whether f_n is a part of the base M_4 or not, the point A lies on its associated quadric. In S_3 the two generators of the associated quadric which pass through A generate a ruled surface of order $2n+3$ as A moves along f_n. This R_{2n+3} is a principal surface in the transformation.

(b) Two pencils of quadric primals give a T_{2n+4} with an inverse of the same order. The properties of this transformation may be found in a manner similar to that of (a).

5. *Case* C. When a pencil of cubic primals is employed, it is

necessary for a point of f_n to be a double point on its associated primal. Then every other point of f_n is a simple point of the primal. The transformation is possible only when f_n is of order $n=1, 2, 3$. The discussion is given for S_3, but the general work may be extended to S_r, $r>3$. The author has made the analytic extension, but omits it here for the sake of simplicity. The properties of transformations (b) and (c) may be found in a manner similar to that of (a).

(a) A pencil of cubic primals and a pencil of primes in S_3. T_{3n+4}; T_{n+4}^{-1}.

(1) $n=1$. T_7; T_5^{-1}. In S_3, the line f_1 is contained twice in the base of the pencil of cubic surfaces, since the surfaces of the pencil touch at every point of f_1, leaving a residual C_7. C_7 meets f_1 in 4 points. The image of each point of C_7 is a conic. The complete image of C_7 is a surface j_{10}' which contains f_1 counted four times and o_2 counted three times, that is, $C_7 \sim j_{10}'$; $o_2^3 f_1^4$.

The image of f_1 is f_1 itself. There are two points of f_1 through which pass their associated planes on o_2. The images of these two points in the inverse transformation are the cubic surfaces associated with these points respectively.

There are two plane cubic curves formed by the intersection of the two planes through their associated points with the corresponding cubic surfaces. These curves are part of the base of the pencil of cubic surfaces. The six lines on any cubic surface which can be drawn from the associated point A generate a ruled surface of order 8, as A moves along f_1, and this R_8 is a principal surface. Every plane through f_1 intersects R_8 in a residual C_3. The curve C_3 breaks up into the three lines joining in pairs the three points of C_7 not on f_1 in the plane.

Two general homaloids of the transformation meet in a variable C_5. The residual intersection is $C_7^4 f_1^{10} \gamma_3 \delta_3$ where γ_3 and δ_3 are the two plane cubic curves mentioned above.

(2) $n=2$. T_{10}; T_6^{-1}. In S_3, the conic f_2 counts twice in the base of the pencil of cubic surfaces, leaving a residual C_5. The plane of f_2 meets C_5 in 5 points, all on f_2. $C_5 \sim j_{10}'$; o_2^5.

The image of any point P on f_2 is a conic passing through P and K, the intersection of o_2 and the plane of f_2, and through the three points of f_2 through which pass the associated planes on o_2. The totality of these conics is the image of f_2. The images, in the inverse transformation, of the three fundamental points of

f_2 are the corresponding cubic surfaces. The ruled surface, analogous to R_8 of the previous case, is, in this case, R_{11}.

(3) $n=3$. T_{12}; T_7^{-1}. In S_3, f_3 counts twice in the base of the pencil of cubic surfaces, leaving a residual C_3. The curve C_3 breaks up into three lines which are bisecants of f_3. $C_3 \sim j_6'$; o_2^3. The image of f_3 is of order 4, containing o_2 twice. There are 4 points on f_3 through which pass associated planes on o_2. Their images are the corresponding cubic surfaces. The principal ruled surface in this case is R_{14}.

(b) A pencil of cubic primals and a pencil of quadric primals give T_{3n+5}, $n=1, 2, 3$, with the inverse T_{2n+5}^{-1}.

(c) Two pencils of cubic primals give T_{3n+6}, $n=1, 2, 3$, with an inverse of the same order.

6. *Case* D. If f_n is taken as a line f_1, it can be of any multiplicity on a surface of any order. Consider a pencil of primals of order s on o_1, and a pencil of primals of order t on o_2. Take f_1 such that a primal on o_1 has an $(s-1)$-fold point at its associated point of f_1 and a primal on o_2 has a $(t-1)$-fold point at its associated point. Then a general point of f_1 is $(s-2)$-fold on every primal of the first pencil, and $(t-2)$-fold on every primal of the second pencil. We obtain a T_{2s+t} with the inverse T_{2t+s}^{-1}.

The locus of invariant points is, as before, the locus of intersection of corresponding elements of the two pencils. In S_3, the bases of the two pencils do not intersect. In S_r, $r>3$, they do, and any point on the common part is invariant under the transformation.

In S_3, the residual part of the base is C_{3s-2} meeting f_1 in $3s-5$ points. The image of C_{3s-2} is of order 6 plus the multiplicity of f_1. If the elements of the second pencil are planes, $C_{3s-2} \sim j_{3s+1}'$; $o_2^3 f_1^{3s-5}$. If the elements of the second pencil are surfaces of order t, the cone joining C_{3s-2} to a point A on f_1 is met by the associated surface of the second pencil in a conical curve of order $(3s-2)t$, the image of C_{3s-2} for the particular surface associated with A.

In S_3, if f_1 is an $(s-1)$-fold line on the first base, the residual part of the base is C_{2s-1}. There is one point of C_{2s-1} not on f_1 in every plane through f_1. The image of C_{2s-1} is of order 2 plus the multiplicity of f_1. If the elements of the second pencil are planes, $C_{2s-1} \sim j_{2s}'$; o_2. If the elements of the second pencil are

surfaces of order t, the image of C_{2s-1} for a particular surface is a conical curve of order $(2s-1)t$.

In S_r, the image of any point on the base M_{α} or M_{α} is a conic. The lines joining points of M to a particular point of f_1 form a conical primal with a point vertex. It is met by the corresponding primal of the second pencil in a manifold $M^{r-2}_{(3s-2)t}$ or $M^{r-2}_{(2s-1)t}$ according as there is contact or not.

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NOTE ON SOME EQUATIONS WITHOUT AFFECT*

BY SAUNDERS MACLANE

A numerical equation of degree greater than 4 certainly cannot be solved by radicals if it is "without affect"; that is, if its Galois group is the symmetric group. Hence it is of interest to construct explicitly such equations. A number of such constructions have been developed,† many of them intrinsically related to certain prime-ideal decompositions. Hence the Newton polygon construction for prime ideals and the related Eisenstein irreducibility criterion are relevant, and can be used systematically to give new proofs for several known constructions (Theorem 2) and for some new equations without affect (Theorems 1 and 2 and generalizations). The advantages lie in the uniform procedure and in the ease of the explicit construction of Theorem 1.

THEOREM 1. *Let p, q, and r be rational primes and construct*

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n, \qquad (n \geqq 4), \tag{1}$$

with rational integral coefficients a_i such that: (I) *each a_i is divisible by r, but a_n is not divisible by r^2*; (II) *each a_i is divisible by q, and a_n but not a_{n-1} is divisible by q^2*; (III) *the highest power e_i such that a_i is divisible by p^{e_i} satisfies*

$$e_1 \geqq 1, \qquad e_2 = 1, \qquad e_3 \geqq 2, \qquad e_i - e_{i-1} > e_{i-1} - e_{i-2}, \tag{2}$$

* Presented to the Society, April 10, 1936, and subsequently extended.

† Ph. Furtwängler, *Ueber Kriterien für irreduzible und für primitive Gleichungen und über die Aufstellung affektfreier Gleichungen*, Mathematische Annalen, vol. 85 (1922), pp. 34–40.

*for $i=4, 5, \cdots, n$. Then the Galois group of $f(x)$, considered as a permutation group G on the roots of $f(x)$, is the symmetric group.**

The assumption I of the theorem is exactly the condition under which the *Eisenstein irreducibility Theorem*† asserts the irreducibility of $f(x)$. Hence the group G is transitive.

In the field $R(\alpha)$, obtained by adjoining to the rational field R a root α of $f(x)$, certain prime ideal factors of a prime q may be found thus: Plot in a plane the points $A_i=(i, d_i)$, where d_i is the exact power to which a_i is divisible by q. The convex broken line which joins $A_0=(0, 0)$ to A_n, which has vertices only at certain of the points A_i, and which passes above none of these points is the *Newton Polygon* of $f(x)$. If a side A_iA_j of this polygon has a positive slope $e/(j-i)$, with e prime to $(j-i)$, then q is known to have‡ a corresponding prime factor Q in $R(\alpha)$ such that q is divisible by Q^{j-i} and no higher power, and α is divisible by Q^e and no higher power.

By hypothesis II the polygon for q has two sides A_0A_{n-1} and $A_{n-1}A_n$. If Q is the prime ideal corresponding to the second side, then $q\equiv 0$ and $\alpha\equiv 0 \pmod{Q}$, $q\not\equiv 0 \pmod{Q^2}$. Hence the equation

$$f_1(x) = f(x)/(x-\alpha) = x^{n-1} + b_1x^{n-2} + b_2x^{n-3} + \cdots + b_{n-1} \tag{3}$$

has coefficients b_i in the field $R(\alpha)$,

$$b_i = \alpha^i + a_1\alpha^{i-1} + a_2\alpha^{i-2} + \cdots + a_i, \qquad (i = 1, \cdots, n-1), \tag{4}$$

whence, by II, $b_i\equiv 0 \pmod{Q}$. But $a_{n-1}\not\equiv 0 \pmod{Q^2}$, so that $b_{n-1}\not\equiv 0 \pmod{Q^2}$. Hence the Eisenstein criterion for the prime ideal Q in $R(\alpha)$ applies to $f_1(x)$ and shows it irreducible. Therefore the subgroup of G corresponding to the field $R(\alpha)$ (that is,

* This construction avoids the Cauchy-Sylow Theorem and the Tchebichef Theorem on the existence of primes between $n/2$ and n, thus simplifying a similar method due to M. Bauer, *Ueber Gleichungen ohne Affekt*, Journal für Mathematik, vol. 132 (1907), pp. 33–35.

† B. L. van der Waerden, *Moderne Algebra*, vol. 1, p. 77.

‡ O. Ore, *Newtonsche Polygone in der Theorie der algebraischen Körper*, Mathematische Annalen, vol. 99 (1928), pp. 84–117.

the subgroup leaving $\alpha=\alpha_1$ fixed) is transitive* on the remaining roots $\alpha_2, \cdots, \alpha_n$.

To decompose p in $R(\alpha)$, form the Newton polygon for the points $B_0=(0, 0)$ and $B_i=(i, e_i)$ representing the powers p^{e_i} in a_i. By III this polygon has the sides $B_0B_1, B_2B_3, \cdots, B_{n-1}B_n$. The last side $B_{n-1}B_n$ has a slope e_n-e_{n-1} and so corresponds to a prime ideal P with $p\equiv 0 \pmod{P}$, $p\not\equiv 0 \pmod{P^2}$, $\alpha\equiv 0 \pmod{P^{e_n-e_{n-1}}}$. By (2) and (4), $b_1=\alpha+a_1\equiv 0 \pmod{P}$. Furthermore, for any $i>1$, P^{e_i} is the highest power of P dividing b_i, for in the expression (4) for b_i the first term is divisible by a higher power of P than the third term, while by (2) any other term $a_j\alpha^{i-j}$ is divisible by a higher power of P than the subsequent term $a_{j+1}\alpha^{i-j-1}$. Thus b_i is divisible by the power P^{e_i} dividing the last term, and $f_1(x)$ satisfies with respect to P the analog of III. The argument can be repeated on $f_2(x)=f_1(x)/(x-\alpha_2)$, and so on. By choosing each time a prime ideal corresponding to the steepest remaining side of the polygon, we finally obtain a prime ideal $\mathfrak{p}$ in the field $K=R(\alpha_1, \cdots, \alpha_{n-2})$ such that the remaining factor of $f(x)$,

$$f_{n-2}(x) = (x-\alpha_{n-1})(x-\alpha_n) = x^2 + c_1x + c_2,$$

has coefficients c_1 and c_2 in K divisible by $\mathfrak{p}^{e_1}$ and $\mathfrak{p}^{e_2}$, respectively. Since $e_1\geqq e_2=1$, $f_{n-2}(x)$ is irreducible by the Eisenstein criterion. The group of $f_{n-2}(x)$ over K is therefore transitive, so that G contains a transposition (α_{n-1}, α_n). This fact, coupled with the transitivity of G and of the subgroup of G leaving $\alpha_1=\alpha$ fixed, suffice† to make G the symmetric group.

Conditions I, II, and III of this theorem specify essentially that r be a power of a prime ideal in $R(\alpha)$, that a factor of q in $R(\alpha_1)$ be a power of a prime ideal in $R(\alpha_1, \alpha_2)$, that a factor of p in K be a power of a prime ideal in $K(\alpha_{n-1})$. Other conditions can have the same ideal-theoretic effect.

THEOREM 2. *Theorem* 1 *remains valid when* I *is replaced by*

* Therefore any irreducible equation satisfying II is primitive. This proves a primitivity criterion due to Furtwängler, op. cit., p. 37.

† The proof is essentially that in van der Waerden, op. cit., p. 191. If p, q, and r are sufficiently large, Theorem 1 could also be established by a theorem of Wegner connecting the prime decomposition and the Galois group; U. Wegner, *Zur Theorie der affektlosen Gleichungen*, Mathematische Annalen, vol. 111 (1935), pp. 738–742.

one of the following conditions; (Ia) $f(x)$ *is irreducible* (mod r); (Ib) r *is a power of a prime ideal in the ring of polynomials* $R[x]$ modulo $f(x)$. *Likewise* II *or* III *may be replaced respectively by the alternatives*

(IIa) $$a_1 \not\equiv 0 \pmod{q},\quad a_2 \equiv a_3 \equiv \cdots \equiv a_n \equiv 0 \pmod{q},\quad a_n \not\equiv 0 \pmod{q^2};$$

(IIb) $$f(x) \equiv \phi(x)(x-c) \pmod{q}, \qquad (\phi(x) \text{ irreducible mod } q);$$

(IIIa) $$f(x) \equiv x^2\psi_1(x)\cdots\psi_t(x) \pmod{p},\quad a_n \not\equiv 0 \pmod{p^2},$$

where x *and the* $\psi_i(x)$ *are distinct irreducible polynomials* (mod p);

(IIIb) $$f(x) \equiv \phi(x)\psi_1(x)\cdots\psi_t(x) \pmod{p},$$

where the $\psi_i(x)$ *are distinct irreducible polynomials of odd degree* (mod p), *and* $\phi(x)$ *is irreducible and of degree two* (mod p).

Conditions Ia, IIb, and IIIb combined give a construction for equations without affect due essentially to Bauer.* On the other hand, conditions I, IIa, and IIIa, with $t=n-2$, generalize a construction of Perron,† who avoided group theoretic arguments by using conditions on $n-1$ primes instead of on three primes, p, q, and r. Other alternatives to conditions I, II, and III are possible.

Proof. The alternative conditions will give the same properties of the group G as did the original I, II, and III. Condition Ia immediately implies the irreducibility of $f(x)$ and hence the transitivity of G. Condition Ib, by a generalization of the Eisenstein criterion, yields‡ the same conclusion.

We now use another prime ideal construction. If $f(x)$ has a non-multiple irreducible factor $\phi(x)$ (mod q), then there is a prime ideal $Q=(q, \phi(\alpha))$ in $R(\alpha)$ which divides q to the first power only.§ In particular, condition IIa gives $f(x) \equiv x^{n-1}(x+a_1)$

* M. Bauer, *Ganzzählige Gleichungen ohne Affekt*, Mathematische Annalen, vol. 64 (1907), pp. 325–327. Also in van der Waerden, op. cit., p. 191.

† O. Perron, *Ueber Gleichungen ohne Affekt*, Sitzungsberichte der Heidelberger Akademie, 1923, No. 3.

‡ S. MacLane, *The ideal-decomposition of rational primes in terms of absolute values*, Proceedings of the National Academy of Sciences, vol. 21 (1935), pp. 663–667.

§ O. Ore, loc. cit., Theorem 1, p. 99.

(mod q) and hence $aQ=(q, \alpha+a_1)$. Thus $\alpha+a_1$ and, by IIa, $a_2, a_3, \cdots, a_n$ are divisible by Q. Hence

$$b_i = \alpha^{i-1}(\alpha + a_1) + a_2\alpha^{i-2} + a_3\alpha^{i-3} + \cdots + a_i \equiv 0 \pmod{Q}, \qquad (i = 1, 2, \cdots, n-1).$$

Since $b_{n-1}\alpha=a_n$ and $a_n\not\equiv 0 \pmod{q^2}$, we have $b_{n-1}\not\equiv 0 \pmod{Q^2}$. Consequently $f_1(x)$ in (3) satisfies the Eisenstein irreducibility criterion with respect to Q, and G is doubly transitive as before.

Condition IIb gives a prime ideal $Q=(q, \alpha-c)$ in $R(\alpha)$, with $f_1(x)\equiv\phi(x) \pmod{Q}$. But Q is of first degree, so that the residue-class field of integers (mod Q) is identical to the field of integers (mod q), and $\phi(x)$ must thus remain irreducible (mod Q). Hence $f_1(x)$ is irreducible.

From condition IIIa we prove by induction that the group G contains a transposition. If $n=2, f(x)=x^2+a_1x+a_2\equiv x^2 \pmod{p}$, $a_2\not\equiv 0 \pmod{p^2}$, $f(x)$ is irreducible by Eisenstein's criterion and its group contains a transposition interchanging the roots. For $n>2$, let $h(x)$ be the irreducible factor of $f(x)$ divisible by $\psi_1(x)$ (mod p) and let α be a root of $h(x)$. Corresponding to ψ_1 there is a prime ideal $P=(p, \psi_1(\alpha))$ in $R(\alpha)$. The remaining equation $f_1(x)=f(x)/(x-\alpha)$ will have a decomposition

$$f_1(x) \equiv x^2\theta_1(x)\cdot\theta_2(x)\cdots\theta_s(x) \pmod{P},$$

where each $\theta_i(x)$ is an irreducible factor of some $\psi_j(x)$ (mod P). If two θ_i's were congruent mod P, then either $\psi_j(x)$ would have a multiple root (mod P), contrary to the fact that an irreducible polynomial over a finite field never has multiple roots, or else two factors $\psi_j(x)$ and $\psi_k(x)$ would have a root in common (mod P), which is impossible. Hence the $\theta_i(x)$ are distinct and distinct from x, while $b_{n-1}\not\equiv 0 \pmod{P^2}$. Thus $f_1(x)$ satisfies the same conditions with respect to P over $R(\alpha)$ as did $f(x)$ with respect to p over R, and the induction assumption, which holds over any algebraic field, yields the required transposition in G.

The treatment of IIIb is similar. Since $\psi_1(x)$ is of odd degree, the prime ideal $P=(p, \psi_1(\alpha))$ has a residue-class field of odd degree over the field of integers mod p. Hence $\phi(x)$, of even degree, must remain irreducible over such a residue-class field. The irreducible factors of $\psi_i(x)$ (mod P) are factors of an irreducible polynomial over a normal extension field, hence have degrees

which are divisors of the odd degree of $\psi_i(x)$. Induction thus applies, and gives, for $n=2$, $f_{n-2}(x) \equiv \phi(x)$, hence irreducible.

This method of constructing equations without affect has the advantage of generality. Theorem 1 is true and the construction can be readily carried out if the a_i are integers of any algebraic field, while p, q, and r are prime ideals in that field. More generally, we can replace the primes by "absolute values." A discrete non-archimedean absolute value* U of a field K is a function such that, for any a in K, $U(a)$ is a rational integer or $+\infty$, with the properties

$$U(ab) = U(a) + U(b);\ U(a+b) \geqq \min(Ua, Ub).$$

Since the Eisenstein irreducibility theorem and the Newton polygon construction generalize to such absolute values, we can state the following theorem.

THEOREM 3. *If the polynomial $f(x)$ in* (1) *has coefficients a_i in any field K, if $f(x)$ is separable and if the conditions* I, II, *and* III *of Theorem* 1 *hold when Ua_i (or Va_i, or Wa_i) takes the place of the power to which a_i is divisible by p (or q, or r), then $f(x)$ is without affect over K.*

For example, if $K = R(y_1, \cdots, y_m)$ is the field of rational functions of m variables $y_1, \cdots, y_m$ with rational coefficients, a particular value U can be defined thus: Consider any irreducible polynomial $\phi(y_1, \cdots, y_m)$, set Ug for any polynomial $g(y_1, \cdots, y_m)$ equal to the highest power to which g is divisible by ϕ, and set $U(g/h) = Ug - Uh$ for any rational function g/h. Hence *Theorem* 1 *remains true if the a_i are polynomials in $y_1, \cdots, y_m$, while p, q, and r are three irreducible polynomials in the same variables.*

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* W. Krull, *Idealtheorie*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 4, no. 3.

SIMILARITY OF MATRICES IN WHICH THE ELEMENTS ARE REAL QUATERNIONS*

BY LOUISE A. WOLF

1. *Introduction.* The purpose of this paper is to give a necessary and sufficient condition that for two matrices, A and B, of which the elements are real quaternions, there exist a non-singular matrix S whose elements are real quaternions such that $SAS^{-1}=B$. The matrices A and B are said to be similar if such a matrix S exists. This paper defines a set of invariant factors for any such matrix, A, in terms of the ranks of certain real polynomials in A.

2. *Definitions and Notations.* If A represents a matrix having m rows and n columns, then A' (read A transpose) is the matrix A with the rows and columns interchanged so that A' has n rows and m columns.

According to E. H. Moore a set of k vectors η_i, each being a matrix having n rows and one column, where $\eta_i' = (y_{i1}, y_{i2}, \cdots, y_{in})$, $(i=1, 2, \cdots, k)$, whose elements y_{ij} are real quaternions, is left linearly dependent with respect to real quaternions if there exists a set of constants q_i, which are real quaternions and not all zero such that $\sum_{i=1}^{i=k} q_i y_{ij}=0$, $(j=1, 2, \cdots, n)$. If no such set of real quaternions, q_i, exists except $q_i \equiv 0$, the vectors η_i are said to be left linearly independent. Similarly the k vectors η_i are right linearly dependent with respect to real quaternions if there exists a set of constants, q_i, which are real quaternions and not all zero, such that $\sum_{i=1}^{i=k} y_{ji} q_i=0$, $(j=1, 2, \cdots, n)$. If no such set of real quaternions, q_i, exists except $q_i \equiv 0$, the vectors η_i are said to be right linearly independent.

Moore considered the columns of a matrix, whose elements are real quaternions, as vectors and defined the rank, r, of such a matrix, S, as the maximum number of columns of S which are right linearly independent with respect to real quaternions. He proved that if a matrix S is of rank r, then r is also the maximum number of rows of S that are left linearly independent with

* Presented to the Society, September 13, 1935.

respect to real quaternions. He further defined a matrix S to be non-singular if the rank equals the order and proved that for every non-singular matrix S there exists a matrix S^{-1} such that $S^{-1}S = SS^{-1} = I$.

3. *A Correspondence.* The algebra of complex quaternions is equivalent to the total complex matric algebra of order 4.* A matrix $\mathcal{A}$ corresponds to a matrix A, $\mathcal{A} \sim A$, where the elements of A are real quaternions, if $\mathcal{A}$ is obtained from A by the replacement of each element of A by a 2×2 matrix obtained from the relations as given by Dickson. The elements of $\mathcal{A}$ are in the complex field and $\mathcal{A}$ is of order $2n$ if A is of order n.

In the following theorems and definition A is a matrix in which the elements are real quaternions. The minimum function of A in the field of real numbers, R, is the polynomial of lowest degree, $g(\lambda)$, in which the coefficients are real and the leading coefficient unity such that $g(A) = 0$. The following theorems can readily be established.†

THEOREM 1. *The minimum function of A in the field R is unique.*

THEOREM 2. *If $\mathcal{A} \sim A$, then the minimum function of A in R is also the minimum function of $\mathcal{A}$ in R.*

THEOREM 3. *If A is non-singular, then $\mathcal{A}$ is non-singular and conversely.*

THEOREM 4. *A necessary and sufficient condition that A which corresponds to $\mathcal{A}$ be similar to B which corresponds to $\mathcal{B}$ is that there exist a non-singular matrix $\mathcal{S}$ with elements in the complex field such that $\mathcal{S}\mathcal{A}\mathcal{S}^{-1} = \mathcal{B}$.*

4. *A Special Type of Matrix with Elements in the Complex Field.* If $\mathcal{A}$ is a matrix obtained from A by the replacement of the real quaternionic elements of A by 2×2 matrices by means of the correspondence of § 3, then $\mathcal{A}$ has the following properties.

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* See L. E. Dickson, *Linear Algebras*, p. 13.

† The proofs omitted in this article may be found in a thesis for the doctorate which is on file under the same title at the University of Wisconsin Library.

THEOREM 5. *The determinant of $\mathcal{A}$ is real.*

THEOREM 6. *The determinants of the $k \times k$ minors of $(\mathcal{A}-\lambda I)$ are real or occur in pairs that are either conjugates of one another or negative conjugates.*

THEOREM 7. *The invariant factors of $(\mathcal{A}-\lambda I)$ are real.*

5. *A Comparison of the Ranks of A and $\mathcal{A}$.* The statements given in this paragraph concerning the rank of matrices with quaternionic elements are taken from the Lectures of E. H. Moore: The vectors η_i, where $\eta_i' = (\delta_{i1}, \delta_{i2}, \cdots, \delta_{in})$, $(i=1, 2, \cdots, n)$, and where $\delta_{ji}=0$ for $j \neq i$ and $\delta_{ii}=1$, form a base with respect to quaternionic coefficients for the vector space of which η is an element, where $\eta' = (q_1, q_2, \cdots, q_n)$, and where the q_i are real quaternions. If a set of k vectors, such as η_i, where $\eta_i' = (q_{i1}, q_{i2}, \cdots, q_{in})$, $(i=1, 2, \cdots, k)$, is right linearly independent with respect to real quaternions, it is always possible to complete the base for the vector space by adding $n-k$ right linearly independent vectors such that all n vectors are right linearly independent with respect to real quaternions. The rank of a matrix A whose elements are real quaternions is $n-s$ if s is the maximum number of right linearly independent vectors η_i such that $A\eta_i=0$, $(i=1, 2, \cdots, s)$.

In considering the rank of the matrix $\mathcal{A}$ which corresponds to the matrix A, it is convenient to consider the columns of these matrices as vectors. Let $A=(\eta_1, \eta_2, \cdots, \eta_n)$, where $\eta_i' = (q_{i1}, q_{i2}, \cdots, q_{in})$ and $q_{ji}=q_{ji0}+q_{ji1}i+q_{ji2}j+q_{ji3}k$.

If $\mathcal{A} \sim A$, then $\mathcal{A}=(\xi_1, \xi_2, \cdots, \xi_{2n-1}, \xi_{2n})$, where $\eta_i \sim (\xi_{2i-1}, \xi_{2i})$ and the elements of ξ_i are in the complex field C.

THEOREM 8. *If the rank of A is k, then the rank of $\mathcal{A}$ is $2k$.*

If A is of order n, then $\mathcal{A}$ is of order $2n$. Suppose that the rank of A is n, then A is non-singular and $\mathcal{A}$ is also non-singular due to the equivalence and hence $\mathcal{A}$ is of rank $2n$. If A is of rank zero, $\eta_i \equiv 0$ and hence $\xi_i \equiv 0$ and $\mathcal{A} \equiv 0$. Therefore $\mathcal{A}$ is of rank zero.

Suppose A is of rank k, where $0<k<n$, then the following consideration shows that $\mathcal{A}$ is at least of rank $2k$. Take k right linearly independent columns of A and designate them as $\eta_1, \eta_2, \cdots, \eta_k$. Take $\xi_1, \xi_2, \cdots, \xi_{2k-1}, \xi_{2k}$ of $\mathcal{A}$ so that η_i

$\sim(\xi_{2i-1}, \xi_{2i})$. Complete a base for the vector space η_i by adding $\eta_{k+1}, \cdots, \eta_n$ to $\eta_1, \eta_2, \cdots, \eta_k$. Let $\xi_{2k+1}, \xi_{2k+2}, \cdots, \xi_{2n-1}, \xi_{2n}$ be the vectors that correspond to $\eta_{k+1}, \cdots, \eta_n$. Then $B=(\eta_1, \cdots, \eta_k, \eta_{k+1}, \cdots, \eta_n)$ and $\mathfrak{B}=(\xi_1, \xi_2, \cdots, \xi_{2k-1}, \xi_{2k}, \xi_{2k+1}, \xi_{2k+2}, \cdots, \xi_{2n-1}, \xi_{2n})$ are both non-singular matrices. B is non-singular by construction and $\mathfrak{B}\sim B$. Thus $\mathfrak{B}$ is of rank $2n$ and $\xi_1, \xi_2, \cdots, \xi_{2k-1}, \xi_{2k}$ are linearly independent. But $\xi_1, \xi_2, \cdots, \xi_{2k-1}, \xi_{2k}$ are columns of $\mathcal{A}$ and hence $\mathcal{A}$ has at least $2k$ linearly independent columns and the rank of $\mathcal{A}$ is equal to or greater than $2k$.

We can also prove as follows that the rank of $\mathcal{A}$ is equal to or less than $2k$. Suppose A is of rank $k=n-r$, where $r<n$. Then there exist r right linearly independent vectors $\eta_i \not\equiv 0$ such that $A\eta_i=0$, $(i=1, 2, \cdots, r)$. Because of the correspondence there are $2r$ right linearly independent vectors $\xi_1, \xi_2, \cdots, \xi_{2r-1}, \xi_{2r}$ such that we have $\mathcal{A}\xi_i=0$, $(i=1, 2, \cdots, 2r)$, where $\eta_i\sim(\xi_{2i-1}, \xi_{2i})$, $(i=1, 2, \cdots, r)$. The linear independence of the $2r$ vectors, ξ_i, was established in the first part of the proof of this theorem. But if $2r$ linearly independent vectors, ξ_i, exist such that $\mathcal{A}\xi_i=0$, then the rank of $\mathcal{A}$ is less than or equal to $2n-2r$; that is, the rank of $\mathcal{A}$ is less than or equal to $2k$. Hence the rank of $\mathcal{A}$ is $2k$ if the rank of A is k.

6. *Similarity of Matrices Whose Elements Are Real Quaternions in Terms of Invariant Factors of These Matrices.* The matrix $\mathcal{A}$, which corresponds to A of order n, is a matrix of order $2n$ with elements in the complex field, C. The irreducible factors of $|\mathcal{A}-\lambda I|$ are of the form $(\lambda-\alpha_j)$, where the α_j are in C. If r_{ij} is the rank of $(\mathcal{A}-\alpha_i I)^j$ and $r_{i0}\equiv 2n$, it can be seen from the classical canonical form for $\mathcal{A}$ that $r_{ij}=2n-\sum_{l=1}^{j} t_i^{(l)}$, where $t_i^{(l)}$ is the number of invariant factors that contain $(\lambda-\alpha_i)^l$. The set of positive integers $r_{i0}-r_{i1}\geqq r_{i1}-r_{i2}\geqq r_{i2}-r_{i3}\geqq \cdots$ is monotonic non-increasing. If $e_{ij}=r_{ij-1}-2r_{ij}+r_{ij+1}$, then $(\lambda-\alpha_i)^j$ appears among the elementary divisors of $(\mathcal{A}-\lambda I)$ a number of times equal to e_{ij} and the first invariant factor is obtained by taking the product of the highest powers of $(\lambda-\alpha_i)$ which appear among the elementary divisors; the second invariant factor is obtained by taking the product of the second highest powers $(\lambda-\alpha_i)$, and so on.

A second method of defining the invariant factors of $(\mathcal{A}-\lambda I)$ will be given which uses the fact that all the invariant factors

of $(\mathcal{A}-\lambda I)$ are in the real field, R. Let $f_1(\lambda), f_2(\lambda), \cdots, f_k(\lambda)$ be the irreducible factors of $|\mathcal{A}-\lambda I| = f(\lambda)$ in R. Let w_{ij} be the rank of $[f_i(\mathcal{A})]^j$ and $w_{i0} \equiv 2n$.

It can easily be established that $w_{ij} = 2n - \sum_{l=1}^{j} \mu_i t_i^{(l)}$, where μ_i is the degree of $f_i(\lambda)$ and $t_i^{(l)}$ is the number of invariant factors of $(\mathcal{A}-\lambda I)$ that are divisible by $[f_i(\lambda)]^l$, where the invariant factors of $(\mathcal{A}-\lambda I)$ are defined in terms of the elementary divisors in the conventional manner. It was previously proved that $f(\lambda)$ is in the real field and hence the irreducible factors $f_i(\lambda)$ of $f(\lambda)$ in R are either linear or quadratic. The two cases are considered separately as follows:

Case 1. Suppose $\mu_i = 1$, then $f_i(\lambda) = \lambda - \alpha_i$, where α_i is real. It is clear that in this case $w_{ij} = r_{ij} = 2n - \sum_{l=1}^{j} t_i^{(l)}$, where $t_i^{(l)}$ is the number of invariant factors divisible by $(\lambda-\alpha_i)^l$ as before.

Case 2. Suppose $\mu_i = 2$, then $f_i(\lambda) = (\lambda-\alpha_i)(\lambda-\bar{\alpha}_i)$, where α_i is complex. All the invariant factors of $(\mathcal{A}-\lambda I)$ are real, hence $(\lambda-\alpha_i)^k$ and $(\lambda-\bar{\alpha}_i)^k$ are divisors of the same $t_i^{(k)}$ invariant factors when the complex field is considered and therefore*

$$w_{ij} = 2n - \sum_{l=1}^{j} t_i^{(l)} - \sum_{l=1}^{j} t_i^{(l)} = 2n - \sum_{l=1}^{j} 2t_i^{(l)}.$$

Thus $w_{ij} = 2n - \sum_{l=1}^{j} \mu_i t_i^{(l)}$, where μ_i is the degree of $f_i(\lambda)$. $w_{i0} - w_{i1} \geqq w_{i1} - w_{i2} \geqq w_{i2} - w_{i3}, \cdots$, is a set of monotonic non-increasing positive integers.

Let $m_{ij} = w_{ij-1} - 2w_{ij} + w_{ij+1}$. Define the characteristic divisors of $(\mathcal{A}-\lambda I)$ to be the polynomials $[f_i(\lambda)]^k$, where $[f_i(\lambda)]^k$ appears a number of times equal to m_{ik}/μ_i.

The invariant factors of $(\mathcal{A}-\lambda I)$ are defined to be in turn the product of the highest powers of the $f_i(\lambda)$ appearing among the characteristic divisors, the product of the second highest powers of the $f_i(\lambda)$ appearing among the characteristic divisors, and so on.

The elementary divisors of $(\mathcal{A}-\lambda I)$ are the distinct linear factors of the characteristic divisors raised to the powers to which the characteristic divisors occur.

THEOREM 9. *The invariant factors of $(\mathcal{A}-\lambda I)$ as defined in terms of characteristic divisors are the same as those defined in the conventional manner.*

* See Menge, this Bulletin, vol. 38 (1932), pp. 88–94.

If $[f_i(\lambda)]^k=(\lambda-\alpha_i)^k$, where α_i is real, $[f_i(\lambda)]^k$ appears m_{ik} times among the characteristic divisors of $(\mathcal{A}-\lambda I)$, where $m_{ik}=w_{ik-1}-2w_{ij}+w_{ik+1}$ and $w_{ik}=2n-\sum_{l=1}^{k}t_i^{(l)}$ and hence $e_{ik}=m_{ik}$, since $e_{ik}=r_{ik-1}-2r_{ik}+r_{ik+1}$ and $r_{ik}=2n-\sum_{l=1}^{k}t_i^{(l)}$.

If $[f_i(\lambda)]^k=(\lambda-\alpha_i)^k(\lambda-\bar{\alpha}_i)^k$, where α_i is complex, $[f_i(\lambda)]^k$ appears $m_{ik}/2$ times among the characteristic divisors of $(\mathcal{A}-\lambda I)$, where $m_{ik}=w_{ik-1}-2w_{ik}+w_{ik+1}$.

In this case $w_{ik}=2n-\sum_{l=1}^{k}2t_i^{(l)}$, and hence $m_{ik}=2t_i^{(k)}-2t_i^{(k+1)}$ if the last two equations are combined. The invariant factors by the original definition are also real and if $(\lambda-\alpha_i)^k$ appears e_{ik} times among the elementary divisors, $(\lambda-\bar{\alpha}_i)^k$ also appears e_{ik} times and thus $f_i(\lambda)=(\lambda-\alpha_i)^k(\lambda-\bar{\alpha}_i)^k$ appears e_{ik} times, where $e_{ik}=r_{ik-1}-2r_{ik}+r_{ik+1}$ and $r_{ik}=2n-\sum_{l=1}^{k}t_i^{(l)}$. Thus $e_{ik}=t_i^{(k)}-t_i^{(k+1)}$, and it is evident that $m_{ik}=2e_{ik}$.

This is sufficient to establish the theorem because the invariant factors defined in terms of characteristic divisors will be the products of the same factors since $(\lambda-\alpha_i)^k$ appears the same number of times among the characteristic divisors as it appears among the elementary divisors.

To consider A whose elements are real quaternions, let $g(\lambda)$ be the minimum function of A in R and let $f_i(\lambda)$ be the distinct irreducible factors of $g(\lambda)$ in R. Let ρ_{ij} be the rank of $[f_i(A)]^j$ and define ρ_{i0} to be n. Then $\rho_{i0}-\rho_{i1}\geqq\rho_{i1}-\rho_{i2}\geqq\rho_{i2}-\rho_{i3}\geqq\cdots$.

Let $\epsilon_{ij}=\rho_{ij-1}-2\rho_{ij}+\rho_{ij+1}$ and let $[f_i(\lambda)]^k$ appear among the characteristic divisors of A a number of times equal to $2\epsilon_{ik}/\mu_i$, where μ_i is the degree of $f_i(\lambda)$.

The invariant factors of A are defined to be in turn the product of the highest powers of the $f_i(\lambda)$ appearing among the characteristic divisors, the product of the second highest powers of the $f_i(\lambda)$ appearing among the characteristic divisors, $\cdots$, in a manner analogous to the above definition in terms of the elementary divisors.

Theorem 4 proves that A is similar to B if and only if the matrices $\mathcal{A}$ and $\mathcal{B}$ are similar, where $\mathcal{A}\sim A$ and $\mathcal{B}\sim B$. $\mathcal{A}$ is similar to $\mathcal{B}$ if and only if the invariant factors of $(\mathcal{B}-\lambda I)$ and $(\mathcal{A}-\lambda I)$ are the same. The invariant factors of A as defined above are the same as the invariant factors of $(\mathcal{A}-\lambda I)$. This is evident when it is recalled that the rank of $f_i(\mathcal{A})$ is $2k$ if the rank of $f_i(A)$ is k, and hence $m_{ik}=2\epsilon_{ik}$. Furthermore, A and $\mathcal{A}$ have the same minimum function in R, namely, $g(\lambda)$. The distinct

irreducible factors of $f(\lambda) = |\mathcal{A} - \lambda I|$ in R are likewise the distinct irreducible factors of $g(\lambda)$, the minimum function of $\mathcal{A}$ in R. Thus the characteristic divisors of $(\mathcal{A} - \lambda I)$ and A are the same and hence their invariant factors are the same. The same is true of $(\mathcal{B} - \lambda I)$ and B because $\mathcal{B} \sim B$. The following theorem has thus been established.

THEOREM 10. *A is similar to B if and only if the invariant factors of A are the same as the invariant factors of B.*

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THE THEOREM "$p \dashv q . = . pq = p$" AND HUNTINGTON'S RELATION BETWEEN LEWIS'S STRICT IMPLICATION AND BOOLEAN ALGEBRA

BY TANG TSAO-CHEN

In this Bulletin, vol. 40 (1934), p. 729, E. V. Huntington pointed out that the relation called "strict implication" in C. I. Lewis's system of logic can be shown to be substantially equivalent to the relation called subsumption in ordinary Boolean algebra. His main result is as follows:

Whenever we find the formula "$p \dashv q$" asserted, we may thereupon write down the formula "$p = pq$"; and conversely, whenever we find the formula "$p = pq$" established, we may write down that the formula "$p \dashv q$" is asserted.

That is, Huntington's relation is

("$p \dashv q$" is asserted) $\rightleftarrows$ ("$pq = p$" is established).

This relation is not the same as the following theorem:

$$p \dashv q . = . pq = p,$$

where "$=$" takes the meaning of logical equivalence given in Lewis's *Symbolic Logic*.

This theorem being not explicitly mentioned in Lewis's *Symbolic Logic*, I shall prove it here.

Throughout this paper we shall follow Lewis's practice of ignoring the distinction, which is characteristic of Huntington's

paper, between "p" and "p is asserted." Our Theorems 1, 2, 3, 4 correspond to Huntington's 15, 16, 17, 21; and our i corresponds to Huntington's Z^*.

Starting from Lewis's system of postulates, we can establish the following theorems (the references are to *Symbolic Logic* by Lewis and Langford and to theorems here):

1. $p \sim p = q \sim q$

[11.02] $p \sim p \dashv q \sim q.$

$$= . \sim \diamond [(p \sim p) \sim (q \sim q)] \quad (1)$$

[12.11] $\sim \diamond [(p \sim p) \sim (q \sim q)]$

$$= \sim \diamond [(p \sim p) \sim (q \sim q)] \quad (2)$$

[19.57, 12.15] $[(p \sim p) \sim (q \sim q)] = (p \sim p)$ (3)

[(1), (2), (3)] $p \sim p \dashv q \sim q. = . \sim \diamond (p \sim p)$ (4)

[11.02] $p \dashv p. = . \sim \diamond (p \sim p)$ (5)

[(4), (5)] $p \dashv p. = . p \sim p \dashv q \sim q$ (6)

[11.03, 11.2, (6)] $p \dashv p. \dashv . p \sim p \dashv q \sim q$ (7)

[12.1, (7)] $p \sim p \dashv q \sim q$ (8)

[(8)] $q \sim q \dashv p \sim p$ (9)

[(8), (9), 11.03] $p \sim p = q \sim q.$

From this last theorem, if we define 0 by

$$0 = q \sim q,$$

where q is any particular proposition, we obtain the following general theorems containing 0:

2. $p \sim p = 0.$

3. $p0 = 0$

[19.57, Def. of 0] $p0 = 0.$

Let us now define i by the relation*

* This definition of i may be replaced by $i = .0 = 0$, and therefore i means simply the identity $0 = 0$. But, by 5, it may easily be proved that $i = . p = p$.

$$i = \sim \Diamond 0.$$

Then a group of theorems containing i is obtained:

4. $pq \dashv p. = i$

[11.02] $pq \dashv p. = . \sim \Diamond [(pq) \sim p]$ (1)

[12.5] $(pq) \sim p = (qp) \sim p = q(p \sim p)$ (2)

[2., 3., (2)] $(pq) \sim p = 0$ (3)

[(1), (3)] $pq \dashv p. = . \sim \Diamond 0$ (4)

[(4), Def. of i] $pq \dashv p. = i.$

5. $p \dashv p. = i$

[4.] $pp \dashv p. = i$ (1)

[12.7, (1)] $p \dashv p. = i.$

6. $p \dashv q. \dashv i$

[19.6] $p \dashv q. \dashv .p0 \dashv q0$ (1)

[3., (1)] $p \dashv q. \dashv .0 \dashv 0$ (2)

[5.] $0 \dashv 0. = i$ (3)

[(2), (3)] $p \dashv q. \dashv i.$

7. $p \dashv q. = :i.p \dashv q$

[19.6: $|(p \dashv q)|p, i|q, (p \dashv q)|r$]

$p \dashv q. \dashv i: \dashv :.p \dashv q.p \dashv q: \dashv :i.p \dashv q$ (1)

[6., (1)] $p \dashv q.p \dashv q: \dashv :i.p \dashv q$ (2)

[12.7] $p \dashv q. = :p \dashv q.p \dashv q$ (3)

[(2), (3)] $p \dashv q. \dashv :i.p \dashv q$ (4)

[12.17] $i.p \dashv q: \dashv .p \dashv q$ (5)

[(4), (5), 11.03] $p \dashv q. = :i.p \dashv q.$

Lastly, we obtain the required theorem:

8. $p \dashv q. = .pq = p$

$$[19.63] \quad p \supset pq. = .p \supset p.p \supset q \tag{1}$$

$$[(1), 5.] \quad p \supset pq. = .i.p \supset q \tag{2}$$

$$[(2), 7.] \quad p \supset pq. = .p \supset q \tag{3}$$

$$[11.03] \quad pq = p. = :pq \supset p.p \supset pq \tag{4}$$

$$[(4), 4.] \quad pq = p. = :i.p \supset pq \tag{5}$$

$$[(3), (5)] \quad pq = p. = :i.p \supset q \tag{6}$$

$$[(6), 7.] \quad pq = p. = .p \supset q. \tag{7}$$

$$[12.11] \quad p \supset q. = .p \supset q \tag{8}$$

$$[(7), (8)] \quad p \supset q. = .pq = p.$$

Huntington's relation may also be deduced from the above theorem. For, from 8., we have

$$p \supset q. \supset .pq = p \tag{a}$$

$$pq = p. \supset .p \supset q \tag{b}$$

If $p \supset q$ is asserted, then by (a) and the Rule of Inference, $pq = p$ may be asserted. That is,

$$(p \supset q) \text{ is asserted} \rightarrow (pq = p) \text{ is asserted.} \tag{c}$$

Again, if $pq = q$ is asserted, then by (b) and the Rule of Inference, $p \supset q$ may be asserted. That is,

$$(pq = p) \text{ is asserted} \rightarrow (p \supset q) \text{ is asserted.} \tag{d}$$

By (11) on page 730 and (a) and (b) on page 731 of Huntington's paper, our "$(pq = p)$ is asserted" is equivalent to his "$(pq \sim p)$ is asserted," which in turn is equivalent to his "$(pq = p)$ is established." So (c) and (d) together give Huntington's relation,

$$(p \supset q) \text{ is asserted} \rightleftarrows (pq = p) \text{ is established.}$$

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ON THE DETERMINATION OF EARTH CONDUCTIVITY FROM OBSERVED SURFACE POTENTIALS*

BY R. E. LANGER

If a direct current is supplied to the earth through a small electrode, a distribution of electrical potentials over the surrounding surface of the earth is produced. The variation of this potential with the distance from the electrode depends upon the manner in which the conductivity of the earth below the surface varies with the depth, the calculation of the potential when this conductivity is known being possible by familiar methods. From the standpoint of geophysics, however, it is most frequently the converse problem which is important. The surface potential is accessible, and is therefore at least theoretically measurable. From it the unknown subterranean conductivity is to be deduced. The discussions of this problem which are to be found in the geophysical literature have almost without exception been based upon considerations of the trial and error type. The potentials associated with a suitable set of hypothesized conductivities are computed, and by *fitting* observed potentials to these, as best may be, inferences about the unknown conductivity are sought.

A direct method, on the other hand, has been given recently by L. B. Slichter† and the present author.‡ It is assumed (as has generally been done) that the conductivity is a function only of the depth, and the Taylor's series expansion of this function is deduced from the surface potentials. The problem may, therefore, be regarded as solved whenever the conductivity function is one which is represented, at least to the depth in question, either by its Taylor's series directly or by analytic extensions of the same. The present paper is intended to extend the deductions to the case in which either the conductivity

* Presented to the Society, April 10, 1936.

† L. B. Slichter, *The interpretation of the resistivity prospecting method for horizontal structures*, Physics, vol. 4 (1933), p. 307.

‡ R. E. Langer, *An inverse problem in differential equations*, this Bulletin, vol. 39 (1933), p. 814.

or its derivative shows a discontinuity within the depth range to be explored. Such discontinuities are evidently to be associated with stratifications of the earth's crust, and for their analysis the previous method, based as it is upon the use of power series, obviously requires extension. It is to be shown how such extension can be made, namely, how both the location and magnitude of the first discontinuity may be determined.

If the region of the earth in proximity with the electrode is idealized as an infinite half-space, the line of the electrode is a line of symmetry, and the cylindrical coordinates (ρ, x) (radial distance from the electrode, and depth) are clearly advantageous. Referred to them, the potential ϕ satisfies the differential equation

$$(1) \qquad \sigma(x)\left\{\frac{\partial^2\phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\phi}{\partial\rho}+\frac{\partial^2\phi}{\partial x^2}\right\}+\frac{d\sigma(x)}{dx}\cdot\frac{\partial\phi}{\partial x}=0,$$

in which $\sigma(x)$ represents the unknown conductivity.* The analysis of this equation may be made along familiar lines. The substitution $\phi = U(\rho)\cdot u(x)$ leads to a Bessel's equation for the radial component $U(\rho)$, and gives for the depth component $u(x)$ the differential equation

$$(2) \qquad u''+\frac{\sigma'(x)}{\sigma(x)}u'-\lambda^2 u=0,$$

in which λ is a positive parameter. The conditions that the potential be everywhere finite and non-increasing, and that both the potential and the current be continuous, dictate for $U(\rho)$ the choice $J_0(\lambda\rho)$. For $u(x)$ they determine that continuous solution $u_0(x, \lambda)$ of the equation (2), which for every positive λ is a non-increasing function of x, and for which $\sigma(x)u_0'(x, \lambda)$ is continuous. The existence of just one such solution may be inferred from the fact that $\sigma(x)$, by its nature, is positive for all values of x. Finally the condition that $\partial\phi/\partial x$ be zero at the earth's surface, except at the electrode, may be made to yield the formula

$$\phi(\rho, x)=\frac{-c}{2\pi\sigma(0)}\int_0^\infty \frac{\sin\lambda a}{a}J_0(\lambda\rho)\frac{u_0(x,\lambda)}{u_0'(0,\lambda)}d\lambda,$$

* Many details which are here omitted are to be found in the papers cited.

in which c represents the current and a the radius of the electrode. If the function $\Omega(\lambda)$ is now defined in terms of the solution $u_0(x, \lambda)$ specified above, by means of the formula

$$\Omega(\lambda) = \frac{-\lambda u_0(0, \lambda)}{u_0'(0, \lambda)}, \tag{3}$$

the surface potential is clearly given by

$$\phi(\rho, 0) = \frac{c}{2\pi\sigma(0)} \int_0^\infty \Omega(\lambda) \frac{\sin \lambda a}{\lambda a} J_0(\lambda\rho) d\lambda.$$

Since this formula is invertible into

$$\Omega(\lambda) = \frac{2\pi a \lambda^2 \sigma(0)}{c \sin \lambda a} \int_0^\infty \phi(\rho, 0) J_0(\lambda\rho) \rho d\rho,$$

it is clear that a knowledge of the surface potential is co-extensive with a knowledge of $\Omega(\lambda)$. The latter may, therefore, be used in place of the former as a basis for the further considerations.

From the asymptotic expansion of the function $\Omega(\lambda)$ a power series in x which is formally the Taylor's series of $\sigma'(x)/\sigma(x)$ is deducible. This series, together with such analytic extensions of it as may exist, defines a function $s(x)$ over an interval, say $0 \leqq x \leqq H$, and from this in turn a function $\sigma_1(x)$ is determined through the formulas

$$\sigma_1(x) = \begin{cases} \sigma(0) e^{\int_0^x s(x) dx}, & \text{for } 0 \leqq x \leqq H, \\ \sigma_1(H), & \text{for } H < x. \end{cases} \tag{4}$$

The possibility or the feasibility of the computation of $\sigma_1(x)$ may be regarded as the factor limiting the constant H, and H represents the depth to which the "exploration" applies. It will be clear at once that if the true conductivity $\sigma(x)$ is representable over the interval $(0, H)$ by means of power series, it necessarily coincides with $\sigma_1(x)$, and its determination over the interval is in consequence completely covered in the discussions cited. In proceeding, therefore, it will be assumed that the coincidence of $\sigma(x)$ and $\sigma_1(x)$ ceases before the depth H is reached, that is, specifically,

$$\begin{aligned} \sigma(x) &= \sigma_1(x), \qquad \text{for } 0 \leqq x < h < H, \\ \sigma(h+) &= \mu\sigma(h-), \qquad \sigma'(h+) = \nu\sigma'(h-), \end{aligned} \tag{5}$$

the constants μ, ν, being positive and not both unity. The solution $u_0(x, \lambda)$ by virtue of its specification then satisfies in particular the relations

$$(6) \qquad u_0(h-, \lambda) = u_0(h+, \lambda), \qquad u_0'(h-, \lambda) = \mu u_0'(h+, \lambda),$$

and the entire problem crystallizes into the determination of the constants h, μ, and ν.

Let the function $\sigma_1(x)$ be looked upon now as a hypothetical conductivity. Then, in the manner outlined above it is associated with the function

$$(7) \qquad \Omega_1(\lambda) = \frac{-\lambda v_0(0, \lambda)}{v_0'(0, \lambda)},$$

with $v_0(x, \lambda)$ representing the non-increasing solution of the differential equation

$$(8) \qquad v'' + \frac{\sigma_1'(x)}{\sigma_1(x)} v' - \lambda^2 v = 0.$$

Since $\sigma_1(x)$ is continuous, the same is to be the case with $v_0(x, \lambda)$ and $v_0'(x, \lambda)$, and since $\sigma_1(x)$ is known the function (7) is computable.

Let $v_1(x, \lambda)$, $v_2(x, \lambda)$ designate the principal solutions of the differential equation (8) relative to the origin, that is, those for which

$$v_1(0, \lambda) = 0, \qquad v_1'(0, \lambda) = 1,$$
$$v_2(0, \lambda) = 1, \qquad v_2'(0, \lambda) = 0.$$

Then the functions (3) and (7) are expressible in the forms

$$(9) \qquad \Omega(\lambda) = \frac{\lambda W(u_0, v_1)}{W(u_0, v_2)}, \qquad \Omega_1(\lambda) = \frac{\lambda W(v_0, v_1)}{W(v_0, v_2)},$$

in which W denotes the Wronskian, that is,

$$W(f, \psi) = f(x)\psi'(x) - f'(x)\psi(x),$$

and in which any x on the range $(0, h)$ will serve in the evaluation of $\Omega(\lambda)$, while any x on $(0, H)$ serves in the case of $\Omega_1(\lambda)$. To verify these assertions it is necessary only to observe: firstly, that $v_1(x, \lambda)$, $v_2(x, \lambda)$ are also solutions of the differential equa-

tion (2) when x is on $(0, h)$; secondly, that the formulas (9) are obviously correct when the right-hand members are evaluated at the origin; and lastly, that the ratios of Wronskians involved are constants as to x on the intervals stated. Since

$$W(v_2, v_1) = \frac{\sigma(0)}{\sigma_1(x)} \quad \text{on } (0, H),$$

it is found that

$$\Omega_1(\lambda) - \Omega(\lambda) = \frac{\lambda\sigma(0)W(u_0, v_0)}{\sigma(h-)W(u_0, v_2)W(v_0, v_2)}\Bigg]_{x=h-}. \tag{10}$$

The evaluation of the right-hand member of this relation hinges now upon a determination of the functional forms of the several solutions which are involved.

When λ is large and x is on an interval on which $\sigma(x)$ is continuous the solutions of the differential equation (2) are known to admit of asymptotic representations with respect to λ. Specifically they may be represented in the form

$$u_j(x, \lambda) = \alpha_j F_+(x, \sigma) + \beta_j F_-(x, \sigma),$$

in which α_j, β_j are appropriate constants, while

$$F_\pm(x, \sigma) = \left[\frac{\sigma(0)}{\sigma(x)}\right]^{1/2} e^{\pm\lambda x}\left\{1 \pm \frac{\tau(x)}{\lambda} + O\left(\frac{1}{\lambda^2}\right)\right\},$$

with

$$\tau(x) = \int_0^x \left\{\frac{\sigma''}{4\sigma'} - \frac{\sigma'^2}{8\sigma^2}\right\} dx.$$

Moreover,

$$F_\pm(0, \sigma) = 1,$$

$$F_\pm'(x, \sigma) = \left\{\pm\lambda - \frac{\sigma'(x)}{2\sigma(x)} + O\left(\frac{1}{\lambda}\right)\right\} F_\pm(x, \sigma).$$

Under corresponding conditions the solutions of the differential equation (8) may, of course, be similarly written

$$v_j(x, \lambda) = \gamma_j F_+(x, \sigma_1) + \delta_j F_-(x, \sigma_1).$$

Consider now any interval, $h \leqq x \leqq h+\epsilon$, on which $\sigma(x)$ is continuous. On this, as on any interval, $u_0(x, \lambda)$ is non-increasing. Since $F_+(x, \sigma)$ is an increasing function, and since it dominates the function $F_-(x, \sigma)$ for any $x>0$, it must be inferred that the coefficient α_0 is sufficiently small to make the term in question asymptotically negligible. Thus

$$u_0(x, \lambda) = F_-(x, \lambda) \quad \text{on} \quad (h, h+\epsilon),$$

and hence

$$u_0'(h+, \lambda) = \left\{-\lambda - \frac{\sigma'(h+)}{2\sigma(h+)} + O\left(\frac{1}{\lambda}\right)\right\} u_0(h+, \lambda).$$

In virtue of the relations (5) and (6) this may be written

$$\text{(11a)} \qquad u_0'(h-, \lambda) = \left\{-\mu\lambda - \frac{\nu\sigma'(h-)}{2\sigma(h-)} + O\left(\frac{1}{\lambda}\right)\right\} u_0(h-, \lambda).$$

An entirely similar consideration, in which it is recalled, however, that $\sigma_1(x)$, $v_0(x, \lambda)$, and $v_0'(x, \lambda)$ are continuous, leads to the corresponding formula

$$\text{(11b)} \qquad v_0'(h-, \lambda) = \left\{-\lambda - \frac{\sigma'(h-)}{2\sigma(h-)} + O\left(\frac{1}{\lambda}\right)\right\} v_0(h-, \lambda).$$

Finally the relation

$$v_2(x, \lambda) = \frac{1}{2}\left\{1 + \frac{\sigma'(0)}{2\lambda\sigma(0)} + O\left(\frac{1}{\lambda^2}\right)\right\} F_+(x, \sigma_1) + \frac{1}{2}\left\{1 - \frac{\sigma'(0)}{2\lambda\sigma(0)} + O\left(\frac{1}{\lambda^2}\right)\right\} F_-(x, \sigma_1),$$

is directly verifiable on the basis of the definition of $v_2(x, \lambda)$. However, the second term of this expression is asymptotically negligible in the presence of the first. It follows that

$$\text{(11c)} \qquad \begin{aligned} v_2(h-, \lambda) &= \frac{1}{2}\left[\frac{\sigma(0)}{\sigma(h-)}\right]^{1/2}\left\{1 + \frac{1}{\lambda}\left[\tau(h) + \frac{\sigma'(0)}{2\sigma(0)}\right] + O\left(\frac{1}{\lambda^2}\right)\right\} e^{\lambda h}, \\ v_2'(h-, \lambda) &= \left\{\lambda - \frac{\sigma'(h-)}{2\sigma(h-)} + O\left(\frac{1}{\lambda}\right)\right\} v_2(h-, \lambda). \end{aligned}$$

With the results (11a), (11b), and (11c) at hand, the evaluation of the right-hand member of the relation (10) is now immediate. It is found thus that

$$\Omega_1(\lambda) - \Omega(\lambda) = \frac{2(A-1)e^{-2h\lambda}}{(A+1)(B+1)},$$

where

$$A = \mu + (\nu - 1)\frac{\sigma'(h-)}{2\lambda\sigma(h-)} + O\left(\frac{1}{\lambda^2}\right),$$

and

$$B = \frac{1}{\lambda}\left[\tau(h) + \frac{\sigma'(0)}{\sigma(0)}\right] + O\left(\frac{1}{\lambda^2}\right),$$

a formula which is more conveniently written according as $\mu \neq 1$ or $\mu = 1$ in the appropriate one of the following forms. That is, if $\mu \neq 1$,

$$\log\left\{\frac{1}{2}[\Omega_1(\lambda) - \Omega(\lambda)]\right\} = -2h\lambda + \log\left(\frac{\mu-1}{\mu+1}\right) + \frac{1}{\lambda}\left[\frac{(\nu-1)\sigma'(h-)}{(\mu^2-1)\sigma(h-)} - \frac{\sigma'(0)}{\sigma(0)} - \tau(h)\right] - O\left(\frac{1}{\lambda^2}\right), \tag{12}$$

and if $\mu = 1$,

$$\log\left\{\frac{\lambda}{2}[\Omega_1(\lambda) - \Omega(\lambda)]\right\} = -2h\lambda + \log\left[\frac{(\nu-1)\sigma'(h-)}{4\sigma(h-)}\right] + O\left(\frac{1}{\lambda}\right). \tag{13}$$

The analysis of these results may now be outlined very simply. The function $\Omega(\lambda)$ is to be regarded as known from the experimentally determined surface potentials. From it the functions $\sigma_1(x)$ and $\Omega_1(\lambda)$ are successively computable. The left-hand members of the relations (12) and (13) are, therefore, at hand. Let the values of (12) be plotted against λ. Then if the resulting graph shows a rectilinear asymptote, it follows from the significance of the respective terms on the right of (12) that the slope and intercept of this asymptote are respectively equal to $-2h$ and to $\log[(\mu-1)/(\mu+1)]$. Hence the graph serves to determine the constants h and μ. An analysis of the distance between

the graph and its asymptote would clearly give the value of ν. If, on the other hand, the graph in question shows no asymptote, that fact signifies that $\mu=1$. In that case the left member of (13) is to be plotted. Since ν is now not equal to unity this graph will show an asymptote, the slope and intercept of which determine h and ν.

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A SERIES OF INVOLUTORIAL CREMONA SPACE TRANSFORMATIONS DEFINED BY A PENCIL OF RULED CUBIC SURFACES*

BY AMOS BLACK

1. *Introduction.* A series of involutorial Cremona transformations of space were defined by Snyder† by means of a correspondence between the surfaces of a pencil of ruled surfaces and the points of a rational curve, called the director curve. The director curve was a part of the basis of the pencil and two of the chief characteristics of the transformation were: one of the principal surfaces was a ruled surface R, all of whose generators were parasitic lines; all the tangent planes of the surfaces of the homaloidal web along a certain curve were fixed, being determined by the surface R.

In this paper we shall define a series of transformations by means of a correspondence between the surfaces of a pencil of ruled cubic surfaces and the points of certain rational curves. The director curve is not part of the basis of the pencil; one of the principal surfaces is a ruled surface R, all of whose generators are parasitic lines; all the surfaces of the homaloidal web have fixed tangent planes along a certain curve, but none of the fixed planes are determined by R.

2. *Definition of the Transformation.* Given a pencil of ruled cubic surfaces $|F_3|$ whose basis curve consists of a double line d

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† Virgil Snyder, *On a series of involutorial Cremona transformations of space defined by a pencil of ruled surfaces*, Transactions of this Society, vol. 35 (1933), pp. 341–347.

and a rational space quintic curve δ_5 which has d for a quadrisecant; also, given a rational space curve r_m, of order m, which meets d and δ_5 in $m-1$ and $m+1$ points, respectively; then a general surface F_3 of the pencil meets r_m in $3m-1$ fixed points and in just one general point O. Thus there is a (1, 1) correspondence between the surfaces of $|F_3|$ and the points O of r_m. We define $F_3 : O$ and O as associated surface and point.

A point $P(y)$ in space will determine a unique surface F_3 of the pencil, hence a unique point $O(z)$ on r_m. The line PO meets F_3 in P, O, and one other point $P'(x)$, which is defined as the image of P. Then on each line of the complex of lines meeting r_m lies a pair of points P, P' in involution.

We shall discuss the case where $m=2$ in detail, and then indicate results for m general.

3. *Equations of the Transformation.* Let the equations of d and r_2 be

$$d: \quad x_3 = 0, \quad x_4 = 0, \tag{1}$$

$$r_2: \quad x_1x_4 - x_3^2 = 0, \quad x_2 = 0. \tag{2}$$

Then $D = [r_2, d] = (1, 0, 0, 0)$, and any point on r_2 can be represented by

$$O(z) = O(z_1, z_2, z_3, z_4) = (\lambda^2, 0, \lambda\mu, \mu^2). \tag{3}$$

Let the pencil of surfaces $|F_3| : d^2\delta_5$ be

$$\mu F(x) - \lambda F'(x) = 0, \tag{4}$$

where λ, μ in (4) is the same as in (3). If $F(O) : O$, then

$$\begin{aligned} F(x) = {} & [a_3'x_1 + a_2x_2 + b_1'x_3 + (b_3' + a_4' - b_1)x_4]x_3^2 \\ & + [b_1x_1 + b_2x_2 + (b_4' + c_1')x_3 + b_4x_4]x_3x_4 \\ & + [c_1x_1 + c_2x_2 + c_4'x_3]x_4^2, \\ F'(x) = {} & [a_2'x_2 + a_3'x_3 + a_4'x_4]x_3^2 \\ & + [b_1'x_1 + b_2'x_2 + b_3'x_3 + b_4'x_4]x_3x_4 \\ & + [c_1'x_1 + c_2'x_2 + (c_1 + b_4)x_3 + c_4'x_4]x_4^2. \end{aligned} \tag{5}$$

Denote by A, B, C and A', B', C' the coefficients (containing x_1 and x_2 only) of x_3^2, x_3x_4, x_4^2 in $F(x)$ and $F'(x)$, respectively, (that is, $A = a_3'x_1 + a_2x_2$); and by α, β, γ the determinants

$$|BC'| = BC' - B'C,\ |AC'|,\ |AB'|;$$

and let

$$\bar{\mu} = A'x_3^2 + B'x_3x_4 + C'x_4^2.$$

Any point on the line joining $P(y)$ to $O(z)$ has coordinates

$$x_i = \rho y_i + \sigma z_i, \qquad (i = 1, 2, 3, 4). \tag{6}$$

The value of ρ/σ for $P'(x)$ is given by

$$\mu F(\rho y + \sigma z) - \lambda F'(\rho y + \sigma z) = 0. \tag{7}$$

Since P is on (4) and O is on r_m, we find

$$\rho[\mu F(z, y) - \lambda F'(z, y)] + \sigma[\mu F(y, z) - \lambda F'(y, z)] = 0, \tag{8}$$

where $F(y, z)$, $F'(y, z)$ are the first polars of $F(y)$, $F'(y)$ with respect to (z), and $F(z, y)$, $F'(z, y)$ are the first polars of $F(z)$, $F'(z)$ with respect to (y), and $\lambda/\mu = F(y)/F'(y)$. Hence

$$\frac{\rho}{\sigma} = -\frac{L_{16}}{K_{11}},$$

where

$$\begin{aligned} L_{16} &= F'(y)F(y, z) - F(y)F'(y, z), \\ K_{11} &= F'(y)F(z, y) - F(y)F'(z, y). \end{aligned} \tag{9}$$

However L_{16} and K_{11} are factorable, and

$$K_{11} = R_4K_7, \qquad L_{16} = R_4F'L_9, \tag{10}$$

where $F' = F'(y)$ and

$$\begin{aligned} R_4 &= y_4F - y_3F', \quad K_7 = \bar{\mu}(\gamma y_3^2 + 2\beta y_3y_4 + \alpha y_4^2) + F_7(y), \\ L_9 &= \bar{\mu}\bar{L}_6 + F_9(y), \\ \bar{L}_6 &= (A\gamma + A'\beta)y_3^3 + (A\beta + A'\alpha + B\gamma + B'\beta)y_3^2y_4 \\ &\quad + (B\beta + B'\alpha + C\gamma + C'\beta)y_3y_4^2 + (C\beta + C'\alpha)y_4^3, \end{aligned} \tag{11}$$

and where $F_7(y)$, $F_9(y)$ are terms containing y_3, y_4 to higher degree than 4 and 5, respectively. On removing the factor R_4, the involutorial transformation is therefore expressed by

$$I_{13}: \quad x_i = y_iF'L_9 - K_7z_i, \qquad (i = 1, 2, 3, 4). \tag{12}$$

It is evident from equations (12) that $K_7 = 0$ is the surface

of invariant points, that $L_9=0$ is the image of r_2, and that $F'=0$ is the image of D.

4. *Images of the Fundamental Elements.* Through a general point O of r_2 passes a generator g of $F(O)$. If P is any point on g, since PO lies on $F(O)$, then P' is indeterminate, and the line g is parasitic. As O traces r_2, the line g generates the surface $R_4=0$. Since all the generators of R_4 are parasitic lines, the surface should appear as a factor of the transformation, as we have seen from (10).

Consider a general point P on d. It remains invariant on every $F(O)$ of the pencil, except when PO is a generator g_O on $F(O)$, hence a generator of R_4. Thus the total image of d is R_4.

Since D is a double point on $F(D)\equiv F'$, every point of F' is transformed into D, hence $D\sim F'$.

The tangent plane of $F(O)$ at O intersects $F(O)$ in a generator g_O and a conic $c_2:O$. Every point of c_2 is transformed into O, hence $O\sim c_2$. As O traces r_2, the c_2 generates the surface $L_9=0$. The point O is invariant in the direction of the tangent of c_2, hence L_9 and K_7 are tangent along r_2.

On every generator of the rational cone with vertex P on δ_5 and standing on r_2 is one point P', the image of P. The locus of P' is a curve c of order two increased by the number of times PO is tangent to $F(O)$ at P. Given an arbitrary point O on r_2, the tangent plane of $F(O)$ at P cuts r_2 in two points M. Given a point M, there is a unique surface for which MP is a tangent at P, thus one point O. This (1, 2) correspondence on r_2 has three coincidences, hence $P\sim c_5:P^3$. However, through P passes one generator g_O of $F(O)$. Hence c_5 is composite and consists of g_O and a $c_4:P^2$. As P traces δ_5 the g_O generates R_4, and c_4 generates a surface $\Delta_{22}=0$. Since R_4 factors out of the transformation, the total image of δ_5 is Δ_{22}. The equations of Δ_{22} are found by finding the image of $F(x)$. We see that

$$F(x):d^2D^2\delta_5\sim FR_4^2F'^2\Delta_{22},$$

where

$$\Delta_{22}=\bar{\mu}^4\overline{L}_6(\beta^2-\alpha\gamma)+F_{22}(y), \tag{13}$$

and $F_{22}(y)$ contains y_3, y_4 to degree higher than eleven. The point P is invariant in the directions of the two tangents of $c_4:P^2$,

hence two sheets of Δ_{22} are tangent to the two sheets of K_7 along δ_8.

5. *Contact Along d.* The two tangent planes of $F'=0$ at any point on d are identical with those of $\mu=0$. It is evident from equations (11), (12), and (13) that each tangent plane of μ (hence of F') is the tangent plane of one sheet of $K_7=0$, one sheet of $L_9=0$, two sheets of a general homaloid $S_{13}=0$, and four sheets of $\Delta_{22}=0$ at every point on d. Also the remaining three sheets of L_9, S_{13}, and Δ_{22} have fixed tangent planes along d, the three planes being identical with the tangent planes of $\bar{L}_6$ along d. The image due to the former contact is F' and the latter is R_4.

Although every tangent plane of L_9, S_{13}, and Δ_{22} is fixed at any point on d, they are all distinct from the tangent planes of R_4 at the point. The tangent planes of F' at D are

$$y_4=0, \quad \pi=b_1' y_3+c_1' y_4=0.$$

At D, one of the three tangent planes of L_9, S_{13}, Δ_{22} coincides with π.

6. *Determination of the Parasitic Lines.* We have seen that every generator of R_4 is a parasitic line. In general none of these generators lies on any other surface of the transformation. We wish to find which of these lines do lie on other surfaces, and any other parasitic lines which may arise.

Given a point O on r_2, the generator g_O on $F(O)$ intersects d in a point L. The surface $F(O)$ has two pinch points M on d. If a point L and a point M coincide, then a generator g_O on $F(O)$ passes through a pinch point of $F(O)$. In this case g_O is not only parasitic but is a line of contact for S_{13}, L_9, K_7, R_4, Δ_{22}.

Given a point L there are two points M. Given a point M, there are two surfaces for which M is a pinch point, hence two points L. This (2, 2) symmetric correspondence on d has two coincidences, hence there are two such parasitic lines. Denote them by $2g$.

The surface F' has two generators passing through D, its associated point. Each is a parasitic line. One lies in the plane $y_4=0$, hence lies on R_4, and factors out of the transformation. The other lies in the plane π and is a parasitic line for the transformation. This line g' is simple on S_{13}, K_7, F', and Δ_{22}.

Since δ_5 is rational and has d for quadrisecant, the surface of trisecants is a ruled surface of order eight, R_8: $\delta_5^3 d^4$. Since r_2 meets d once and δ_5 three times there are three trisecants of δ_5 which meet r_2. If O_1 is a point on r_2 through which passes a trisecant of δ_5, then the trisecant lies on $F(O_1)$ and is parasitic. Thus there are three such parasitic lines, $3g''$. They are simple on S_{13}, L_9, K_7 and triple on Δ_{22}.

7. *Table of Characteristics.* The images of planes and of fundamental elements can now be expressed by the following table:

$$
\begin{aligned}
S_1 &\sim S_{13}: r_2^2 d^{7+2t+2t'+3t''} D^7 \delta_5^4 2g\, g' 3g'', \\
r_2 &\sim L_9: r_2 d^{5+t+t'+3t''} D^5 \delta_5^3 2g 3g'', \\
d &\sim R_4: r_2 d^3 D^3 \delta_5 2g, \\
D &\sim F': d^{2+t+t'} D^2 \delta_5 g', \\
\delta_5 &\sim \Delta_{22}: r_2^4 d^{11+4t+4t'+3t''} D^{11} \delta_5^7 2g\, g' 3g''^3, \\
&\quad K_7: r_2 d^{4+t+t'} D^4 \delta_5^2 2g\, g' 3g'', \\
J_{48} &= L_9 R_4^2 F'^3 \Delta_{22},
\end{aligned}
$$

where the t and t' in the multiplicity of d stands for the contact as given in §5.

Thus the complete intersection of two surfaces of the web may be written

$$[S_{13}, S_{13}'] = C_{13} r_2^4 d^{49+4t+4t'+3t''} \delta_5^{16} 2(2g) g' 3g''.$$

8. *The Transformation for m General.* We replace r_2 by a rational curve r_m, which meets d and δ_5 in $m-1$ and $m+1$ points, respectively, with parametric representation

$$z_i = z_i(\lambda, \mu), \qquad (i = 1, 2, 3, 4), \tag{14}$$

where the z_i are homogeneous functions of degree m in λ, μ. Denote the $m-1$ points of intersection of r_m with d by D_i, and the associated surfaces by $F^{(i)}$. The transformation has the form

$$I_{6m+1}: x_i = y_i F^{(1)} F^{(2)} \cdots F^{(m-1)} L_{3m+3} - K_{3m+1} z_i(F, F'), \qquad (i = 1, 2, 3, 4), \tag{15}$$

where $K_{3m+1} = 0$ is the surface of invariant points, $L_{3m+3} = 0$ is the image of r_m, and $F^{(i)} = 0$ is the image of D_i. The image of d is $R_4 = 0$, as before. The image of a general point P on δ_5 is a curve

$c_{2m}:P^m$. As P traces δ_5, the c_{2m} generates a surface Δ_{12m-2}, the total image of δ_5.

Each of the two tangent planes of each of the $m-1$ surfaces $F^{(i)}$ at each point of d is the tangent plane of one sheet of L_{3m+3}, one sheet of K_{3m+1}, two sheets of a general S_{6m+1}, and four sheets of Δ_{12m-2}. Also the remaining three sheets of L_{3m+3}, S_{6m+1}, and Δ_{12m-2} have fixed tangent planes along d.

The parasitic lines are as follows: $2g$, two generators of R_4 which are simple lines of contact for L, K, S, R, Δ; $(m-1)g'$, $m-1$ lines, one on each $F^{(i)}$, and simple on S, K, Δ; $(m+1)g''$, trisecants of δ_5 meeting r_m, simple on S, L, K and triple on Δ.

The results can be expressed by the following table:

$$S_1 \sim S_{6m+1}:r_m^2 d^{4m-1+2t_i+2t_i+3t'} D_i^{4m-1}\delta_5^{2m}2g(m-1)g'(m+1)g'',$$

$$r_m = L_{3m+3}:r_m d^{2m+1+t_i+t_i+3t'} D_i^{2m+1}\delta_5^{m+1}2g(m+1)g'',$$

$$d \sim R_4:r_m d^3 D_i^3 \delta_5 2g,$$

$$D_j \sim F^{(j)}:d^{2+t_j+t_j} D_j^2 \delta_5 g_j',$$

$$\delta_5 \sim \Delta_{12m-2}:r_m^4 d^{8m-5+4t_i+4t_i+3t'} D_i^{8m-5}\delta_5^{4m-1}2g(m-1)g'(m+1)g''^3$$

$$K_{3m+1}:r_m d^{2m+t_i+t_i} D_i^{2m}\delta_5 2g(m-1)g'(m+1)g'',$$

$$J_{24m} = L_{3m+3}R_4^2 F^{(1)^2}\cdots F^{(m-1)^2}\Delta_{12m-2},$$

where the t_i and t' represent the contact along d, and $i=1, 2, \cdots, m-1$.

The complete intersection of two surfaces of the web is

$$[S_{6m+1}, S'_{6m+1}] = C_{6m+1}r_2^4 d^{(4m-1)^2+(m-1)(4+4)+3}\delta_5^{(2m)^2}2(2g)(m-1)g'(m+1)g''.$$

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